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PROCEEDINGS
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ON THE STABILITY OF THE S_8 MOLECULE AND THE STRUCTURE OF FIBROUS SULFUR

By Linus Pauling

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Communicated June 15, 1949

Sulfur is unique among the elements in consisting in its standard state of octatomic molecules. It is interesting that the stability of the S_8 molecule can be understood by consideration of the forces determining the orientation around the sulfur-sulfur single bond.

Let us consider a molecule such as H_2S_2, in which each sulfur atom forms a single bond with the other sulfur atom and a single bond with another atom (hydrogen), and has in addition two unshared electron pairs in its outer shell. The bonds formed by the sulfur atoms involve mainly $3p$ orbitals, and one of the two unshared outer electron pairs occupies the third $3p$ orbital, the other unshared pair occupying essentially the $4s$ orbital. (A rough theoretical treatment which will be communicated later indicates that the bond orbitals have only about 1% s character.) The dihedral angle between the planes H—S—S and S—S—H is determined principally by the repulsion of $p\pi$ electrons. The repulsion between $p\pi$ electrons is at a maximum when the $p\pi$ orbitals of the two sulfur atoms are in the same plane, and falls to zero when these orbitals are in orthogonal planes. The energy of interaction of an unshared pair of $p\pi$ electrons on one atom and the similar unshared pair on the other atom in a coplanar $p\pi$ orbital is equal to -2α (the negative sign corresponding to repulsion, because α, the exchange integral occurring in the electron-pair bond theory, is itself negative), that between an unshared pair on one atom and a single bonding electron on another is $-\alpha$, and that between two bonding electrons is $-{}^1/_2\alpha$. Hence the total interaction energy of the $p\pi$ electrons in H_2S_2 varies from $-{}^5/_2\alpha$ for a planar configuration of the molecule (dihedral angle γ equal to 0° or 180°) to a minimum of -2α for $\gamma = 90^\circ$. It is then to be predicted that the molecule H_2S_2 has a skew configuration, with the dihedral angle γ equal to approximately 90°. The configuration of this molecule has not been precisely de-

termined, although it is known not to be planar,[1] but the similar molecule hydrogen peroxide, H_2O_2, has been shown to have γ equal to approximately 106° in the crystal formed by hydrogen peroxide and urea[2] and a similar value in the pure substance.[3] (An explanation essentially equivalent to that given above was advanced by Penney and Sutherland to account for the skew configuration of hydrogen peroxide.[4]) Moreover, Guthrie has found, from a careful electron-diffraction investigation of sulfur monochloride, S_2Cl_2, that the dihedral angle in this molecule is equal to 92° ± 12°,[5] and the value of γ in dimethyltrisulfide has been found[6] to be approximately 106°. We accordingly conclude that the normal value of the dihedral angle γ for the sulfur-sulfur bond is in the neighborhood of 100°. The increase over the value 90° given by the simple theoretical treatment may be due to van der Waals repulsion of the non-bonded atoms and to other in-

TABLE 1

VALUES OF THE DIHEDRAL ANGLE IN STAGGERED RING MOLECULES

NUMBER OF ATOMS IN RING	BOND ANGLE						
	90°	95°	100°	105°	110°	115°	120°
4	0°	.....	.....	.....	.....	.....	.....
6	90°	84°31′	77°52′	69°34′	58°41′	42°13′	0°
8	114°28′	111°0′	106°50′	102°6′	96°18′	89°10′	80°7′
10	128°10′	125°34′	122°32′	118°59′	114°48′	109°47′	103°39′
12	137°4′	134°57′	132°31′	129°41′	126°21′	122°24′	117°39′

teractions, such as the electrostatic interactions of multipoles discussed recently by Lassettre and Dean [7]

In the molecule S_8, which has the staggered configuration, the S—S—S bond angle β has been found to be equal to 105° both in the orthorhombic crystal[8] and in the vapor.[9] The dihedral angle γ is not independently variable, but is determined by β; its value is 102°. This value is, according to the foregoing argument, close to the normal value, which minimizes the energy of the X—S—S—X group; and it is our thesis that the S_8 molecule is stable (relative to S_x, with $x \neq 8$) because its configuration leads to this value.

The relation between γ and β for symmetrical staggered rings (with point-group symmetry D_{nh}) containing n atoms (n even) is

$$\sin^2 \frac{\gamma}{2} = \frac{\cos (2\pi/n) + \cos \beta}{1 + \cos \beta}. \qquad (1)$$

In table 1 there are given values of γ for values of β between 90° and 120° and for n between 4 and 12. It is seen that for $\beta = 105°$, the value observed for sulfur, the minimum deviation from the normal value of γ, which we may take as 100°, occurs at $n = 8$, and amounts to about 2° as compared

with 19° for $n = 10$ and 30° for $n = 6$. Since the staggered configurations of the ring provide better approximation to the normal value of γ than do other configurations, this calculation gives an explanation of the stability of S_8 relative to other sulfur molecules. At higher temperatures sulfur vapor contains S_6 and S_2 as well as S_8. There is little doubt that the S_6 molecule is a staggered hexagonal ring. The difference in heat content of S_6 and S_8 has been evaluated from thermochemical data by Bichowsky and Rossini,[10] and found to be equal to 1.25 kcal./mole per sulfur-sulfur bond. The interaction energy of the $p\pi$ electrons is according to simple theory proportional to $\sin^2 \gamma$. On this assumption, and the assumptions that the bond angle β is the same in S_6 as in S_8 and that the normal value of γ is 100°, α can be evaluated as -9.8 kcal./mole. This value for α can be considered to be only a very rough one, inasmuch as other structural factors also without doubt contribute to the difference in energy of S_6 and S_8, but it is worth pointing out that the value is not unreasonable.

The standard state of selenium is a hexagonal crystalline state in which there are infinite spiral molecules of selenium atoms, with Se—Se—Se bond angle $\beta = 105°$. There also exists a less stable red crystalline state of selenium containing Se_8 molecules, and the vapor of selenium consists of Se_8 molecules, Se_6 molecules, and, at higher temperatures, Se_2 molecules. Bichowsky and Rossini give for the difference in energy of Se_6 and Se_8 the value 1.07 kcal./mole of Se—Se bonds, which corresponds to -8.4 kcal./mole for the resonance integral α.

The spiral molecules in the hexagonal crystals of selenium, and also those representing the standard state of tellurium, have a threefold axis. The relation between the dihedral angle γ and the bond angle β for a threefold spiral molecule is $\sin^2 \gamma = {}^1/_2 (1 - 2 \cos \beta)/(1 - \cos \beta)$, corresponding for the values 90°, 95°, 100°, 105°, 110°, 115° and 120° for β to the values 90°, 94°36′, 98°31′, 101°52′, 104°46′, 107°17′ and 109°28′ for γ. It is seen that for $\beta = 105°$, as observed for sulfur and selenium, the value of γ is nearly the same for the threefold spiral as for the 8-membered staggered ring. The greater stability of the crystals containing the threefold spiral for selenium and tellurium is presumably due to the strong interaction between neighboring molecules which is made evident in the semi-metallic properties of the crystals. For tellurium, with bond angle 102°, the value of γ is 100°.

The failure of oxygen to form a molecule such as O_8 is of course due to the greater stability of the multiple bonds in the O_2 molecule; this greater stability of multiple bonds over single bonds is characteristic of the first-row elements.

It is interesting that fibrous sulfur, made by stretching the rubbery form of sulfur obtained on cooling liquid sulfur-μ, seems not to contain spirals with a threefold axis, like the spirals in selenium and tellurium, but to have another structure. Meyer and Go[11] have reported that the unit of struc-

true of fibrous sulfur is monoclinic, with $a_0 = 26.4$ Å, $b_0 = 9.26$ Å, $c_0 = 12.32$ Å and $\beta = 79°15'$, and that it contains 112 atoms. The b axis is the fiber axis. These authors suggested that there are 14 chains with a repeating unit of eight atoms each in the unit, and proposed two alternative planar configurations for the chains. These planar configurations are, however, to be rejected, because they make the S—S—S—S dihedral angle equal to 0°,

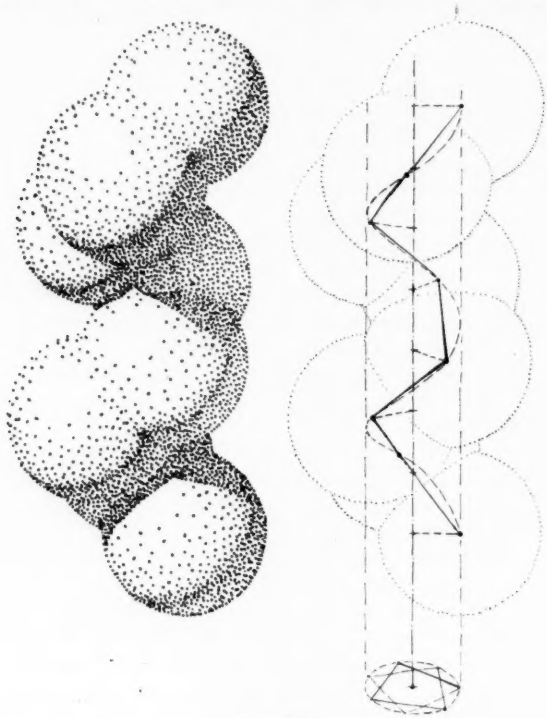

FIGURE 1
The seven-atom two-turn spiral proposed as the repeating unit in the chains of fibrous sulfur.

which corresponds to the minimum stability. It might indeed be expected that the sulfur atoms are arranged in a simple threefold spiral, with two turns in the unit distance b_0, as was tentatively suggested by Huggins.[12] This, however, leads to the value 109° for the S—S—S bond angle, which is unsatisfactory, and, moreover, the reported number of atoms in the unit is not a multiple of 6.

A conclusion about the probable structure can be reached from the assumption that the S—S distance and the S—S—S bond angle have the same values (2.08 Å and 105°) as in S_8. It is found that no spiral making a single turn in the identity distance b_0 is acceptable. A spiral making two turns in b_0 would be predicted, with the above parameters, to have $b_0 = 8.89$ Å for six atoms, 9.12 Å for seven and 8.49 Å for eight. Only the second of these is close enough to the observed value 9.26 Å to be indicated as correct; indeed, increasing the bond angle to 106° leads for the seven-atom two-turn spiral to exactly $b_0 = 9.26$ Å. The dihedral angle for this spiral is 76°, which is intermediate between the value for S_8 and that for S_6, and, through its derivation from the value corresponding to maximum stability, is compatible with the ease of conversion of fibrous sulfur into orthorhombic sulfur. The seven-atom chain unit explains the unusual occurrence of a multiple of seven atoms (112) in the unit of structure; the packing of chains is such as to lead to sixteen chains, parallel to b, per unit. Moreover, the fact that b_0 is 9.26 Å rather than approximately half so large is explained by the seven-atom spiral, which repeats only after seven atoms, whereas the six-atom and eight-atom spirals (with two turns) would repeat after a single turn. It is accordingly very likely that the structural unit of fibrous sulfur is the seven-atom two-turn spiral, shown in figure 1. The geometrical properties of the spiral are such that the distances $b_0 = 9.26 \pm 0.05$ Å and S—S $= 2.08 \pm 0.02$ Å determine the bond angle to be $106° \pm {}^1/_2°$.

* Contribution No. 1296 from the Gates and Crellin Laboratories of Chemistry.

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EVIDENCE FOR A TRYPTOPHANE CYCLE IN NEUROSPORA*

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Communicated by G. W. Beadle, July 15, 1949

The work of Snell[1] on lactic acid bacteria provided the first evidence that anthranilic acid may act as a precursor to tryptophane *in vivo*. Since the strains of bacteria which utilize anthranilic acid in the place of tryptophane also use indole, it was not possible to establish the order in which these two precursors occurred. In 1943 Tatum, Bonner and Beadle[2] found that anthranilic acid was accumulated in the culture medium by a *Neurospora crassa* mutant, strain 10575, which required either tryptophane or indole for growth. It was therefore indicated that anthranilic acid is a precursor to indole in Neurospora.

It has recently been observed in this laboratory that when a medium containing tryptophane is inoculated with wild type or any one of a number of tryptophane-utilizing strains of Neurospora, a blue fluorescence, resembling that of an anthranilic acid solution, is produced. The fluorescence is detectable within a few hours after inoculation, reaches a maximum intensity at 24 to 30 hours and then diminishes until, at 72 hours, the medium displays essentially no fluorescence except in the case of the mutants which are able to use tryptophane but cannot utilize anthranilic acid. Earlier work has already demonstrated that anthranilic acid is formed by the degradation of tryptophane by bacteria[3, 4] and in the rat and other animals.[5] Thus the appearance and subsequent disappearance of fluorescence in Neurospora medium suggested that in Neurospora some of the reactions by which tryptophane is degraded and synthesized may constitute a metabolic cycle. The present experimental evidence supports this conclusion.

An earlier communication[6] from this laboratory postulated 3-hydroxykynurenine as an intermediate in the conversion of tryptophane to nicotinic acid in Neurospora. Through the generosity of Dr. W. Weidel, of the Kaiser Wilhelm Institut für Biochemie, a sample of this compound was made available to the authors and it became possible to establish its relation to the tryptophane cycle in Neurospora. The substance was isolated by Weidel in 1949 from larvae of *Calliphora erythrocephala* as the cn$^+$ substance (precursor of eye pigments in insects).

Experimental.—Mutant Strains.—The qualitative growth responses of the various Neurospora strains considered in this investigation are listed in table 1. Previous descriptions have been made of strains E5212,[7] C86,[7, 8] 10575,[2] C83[8, 9] and 4540.[10]

Disappearance of Tryptophane from the Culture Medium.—Erlenmeyer

flasks (125 ml.) containing 20 ml. of minimal medium[11] supplemented with 400 γ of L-tryptophane were inoculated with conidia from Neurospora strain B1312 and were incubated at 25°C. Absorption spectra of the medium before inoculation and at 8-hour intervals thereafter were determined with a Beckman spectrophotometer. These spectra are shown in Figure 1. Dry weights of mold per flask at the different times were as follows: 16 hours, 0.6 mg.; 24 hours, 3.0 mg.; 32 hours, 9.1 mg. The subsequent dry weight attained at 96 hours was 55 mg. On the basis of the spectra at 280 mμ (the absorption maximum for the indole ring of tryptophane), it is obvious that at least 100 γ of the tryptophane are gone when the growth of mold is only 0.6 mg. By the time 3 mg. of growth has been produced, at least half of the tryptophane has disappeared, and the shape of the absorption spectrum suggests that actually much more than half is gone.

In order to eliminate the possibility that the tryptophane was not simply absorbed from the medium by the small amount of mold present in the early stages of growth, the mold was removed from 32- and 96-hour cultures,

TABLE 1

UTILIZATION OF VARIOUS COMPOUNDS BY *Neurospora* MUTANTS

STRAIN	TYROSINE AND CINNAMIC ACID	PHENYL-ALANINE	ANTHRA-NILIC ACID	INDOLE	TRYPTO-PHANE	KYNU-RENINE	3-HY-DROXY-KYNU-RENINE	3-HY-DROXY-ANTHRA-NILIC ACID	NICO-TINIC ACID
E5212	−	+	−	−	−	−	−	−	−
C86	+	+	+	+	+	+	+	+	+
B1312	−	−	+	+	+	+	−	−	−
10575	−	−	−	+	+	−	−	−	−
C83	−	−	−	−	+	−	−	−	−
E5029	−	−	−	−	−	−	+	+	+
4540	−	−	−	−	−	−	−	−	+

hydrolyzed with 1 *N* NaOH, and the tryptophane content was determined by bioassay with *S. lactis R.*[1] A control sample of tryptophane was subjected to the same treatment. The tryptophane content of the 32-hour culture on a dry-weight basis was found to be 0.3% and that of the 96-hour culture 0.2%. After 32 hours the mold therefore contains approximately 27 γ of tryptophane as opposed to a disappearance of at least 200 γ. Thus it is clearly demonstrated that the tryptophane has not been taken up as such by the mold, but has actually been converted to other compounds.

Products of Tryptophane Degradation.—Flasks of medium supplemented with 400 γ of L-tryptophane per 20 ml. were inoculated with conidia from wild-type strain 8a and from mutants B1312, C86 and 10575, respectively. After 32 hours' incubation at 25° the mold was removed from the flasks, the medium was filtered, the pH was adjusted to 2.5 and the medium was extracted four times with an equal volume of ether. The ether was evaporated on a warm water bath. With each strain the absorption spectrum of

the material in the ether phase was found to be nearly the same as the absorption spectrum of the unextracted medium. Measurements of fluorescence using a Coleman photofluorometer disclosed that approximately 80% of the fluorescence originally present in the unextracted medium went to the ether phase. The residual aqueous phase had an absorption spectrum corresponding to a maximum of 50 γ of tryptophane. Bioassays, using strains of C86 and B1312, showed that from the original 400 γ of tryptophane there remained in the medium after 32 hours' growth the biological equivalent of approximately 250 γ. Of this, approximately 180 γ were extracted with ether. With the unextracted medium, strains 10575 and C83 which

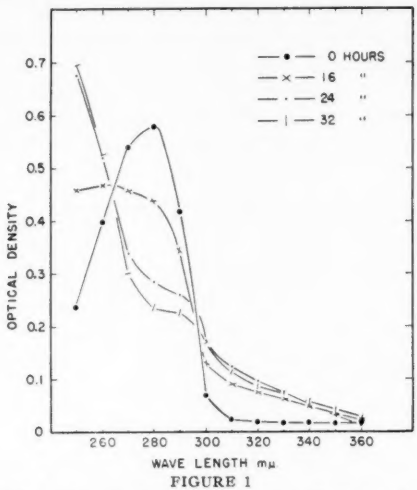

FIGURE 1

Absorption spectra of *Neurospora* medium at various times after inoculation.

do not use anthranilic acid gave growth which was equivalent to that obtained on about 50 γ of tryptophane, but neither of these strains grew on the ether extract. All four of the strains tested gave the same general qualitative and quantitative results as determined by measurements of absorption spectra, fluorescence and biological activity.

Filter paper strip chromatograms were run on the ether extract with water-saturated butanol as the solvent. After development, the chromatograms were examined under ultra-violet light to detect fluorescent areas, and were then cut into sections, the sections were extracted with 0.1 *M* KH_2PO_4 (pH 3.5), and the biological activities of the resulting solutions were determined using B1312 for the assays. It was found that approximately 60% of the activity of the ether phase could be accounted for by the

activity of a chromatogram band which resembled an anthranilic acid band in its position and fluorescence.

In order to obtain enough material for isolation of some of the degradation products, 8 liters of culture medium containing 8 g. of L-tryptophane was inoculated with a heavy inoculum of conidia from wild-type strain 8a. The culture was incubated with forced aeration in a 35° water bath for 19 hours. The mycelium was then filtered off, the pH of the filtrate was adjusted to 2.5, the filtrate was divided into two portions and each was extracted with seven 500-ml. portions of ether. Following evaporation of the ether, the residual solution was absorbed on 29 sheets of filter paper and these sheets were incorporated into a chromatopile[12] consisting of 600 sheets of 9-cm. Whatman No. 1 filter paper. Water-saturated butanol was used as the developer. After 15 hours, every twentieth sheet was removed from the pile and the location of anthranilic acid was determined by absorption spectra of extracts of the sample sheets. The desired section was then removed from the chromatopile, dried in a vacuum desiccator over $CaCl_2$ and extracted with ether in a Soxhlet. The residue from evaporation of the ether was sublimed at 100° *in vacuo* (20 microns Hg). The sublimate was crystallized from water and further purified by resublimation in a temperature gradient at 20 microns pressure. The material obtained in this way melted at 143–144° (cor.). A known sample of anthranilic acid melted at 144–145° (cor.) and a mixture of the two melted at 144–145° (cor.). The neutralization equivalent of the isolated compound was 137 (calculated for anthranilic acid—137). The absorption spectrum, fluorescence and biological activity were identical with that of anthranilic acid, furnishing further proof that the isolated compound was indeed anthranilic acid.

Chromatographic examination of the products of tryptophane degradation by germinating Neurospora has demonstrated the existence of at least two biologically active substances besides anthranilic acid. In addition bio-assay with *L. arabinosus* has shown the production of 4 γ of nicotinic acid from 400 γ of tryptophane in 32 hours.

A number of compounds other than tryptophane were tested to determine whether they were able to give rise to fluorescent degradation products in the medium. Strain C86 was used as the test strain. It was found that strain C86 produced the typical blue fluorescence in the medium when supplied with phenylalanine, indole or kynurenine. No appreciable fluorescence was produced when the medium was supplemented with trans-cinnamic acid, tyrosine, 3-hydroxykynurenine or nicotinamide. A test on 3-hydroxyanthranilic acid appeared also negative but inasmuch as the compound itself had some blue fluorescence the test was not critical. In the cases where fluorescence was produced it was not permanent but reached a maximum and then disappeared, as did the fluorescence produced when C86 was supplied with tryptophane.

Discussion.—The evidence reported here establishes the fact that tryptophane disappears rapidly from culture medium which has been inoculated with wild-type Neurospora strain 8a or mutant strains C86, B1312 or 10575. A large part of the tryptophane is gone before appreciable growth of the mold has occurred. It is also established that anthranilic acid is one of the products of tryptophane breakdown in Neurospora.

The accumulated evidence points clearly to the existence in Neurospora of a metabolic cycle involving, among other compounds, tryptophane and anthranilic acid. A schematic representation of such a cycle is shown in

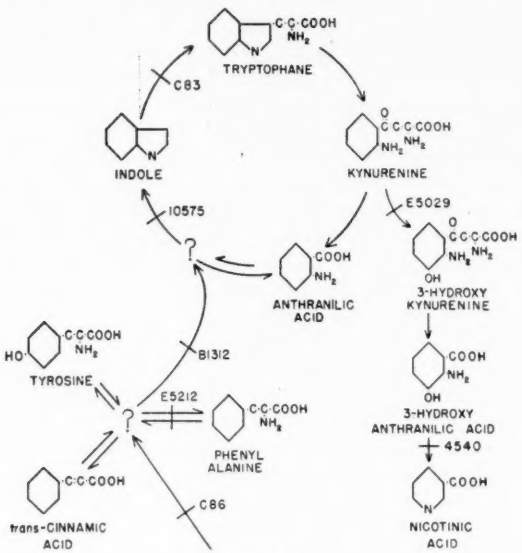

FIGURE 2

Schematic representation of a tryptophane cycle in *Neurospora*.

figure 2. Since strain B1312 is able to utilize anthranilic acid or kynurenine, but not 3-hydroxykynurenine, it is apparent that kynurenine is an intermediate in the formation of anthranilic acid from tryptophane and that 3-hydroxykynurenine is not. In addition the mold will produce blue fluorescence in the medium when supplied with kynurenine, but not when supplied with the hydroxy compound. The reactions by which anthranilic acid is formed from tryptophane must include at least one essentially irreversible step, for strains 10575 and C83 cannot utilize anthranilic acid although they

are able to make it from tryptophane. This is in agreement with the finding by Kotake[5] that rats can make anthranilic acid from tryptophane, but are evidently unable to use anthranilic acid to satisfy their tryptophane requirement.

As previously reported[13] the carboxyl carbon atom of anthranilic acid is lost as CO_2 during metabolism of this substance by Neurospora. It seems most probable that this occurs in the conversion of anthranilic acid to indole since Heidelberger has traced the corresponding carbon of the indole ring of tryptophane to kynurenine and nicotinic acid in rats.[14, 15, 16] For this reason anthranilic acid has tentatively been excluded as an intermediate between phenylalanine and indole in the diagram in figure 2. The accumulation of anthranilic acid by strain 10575[2] is not sufficient reason for placing anthranilic acid immediately before indole since strain C83 also accumulates this substance. It seems more likely that the accumulation of anthranilic acid is due to a reversible reaction between this compound and an unknown intermediate with anthranilic acid formation being favored as shown in the diagram. The accumulation of anthranilic acid during tryptophane degradation by wild type is in agreement with this conclusion. This problem requires further elucidation before the position of anthranilic acid in the cycle can be established with certainty. The number or nature of all the intermediates in the cycle is not known, but as stated above, there are at least three biologically active substances other than anthranilic acid produced as the result of tryptophane degradation. The task of isolating and identifying these intermediates and side products is in progress.

It is clear from the experimental evidence that at least one-fourth of the tryptophane degraded by the mold is converted to substances lacking biological activity for the tryptophane series of mutants. It is not yet clear, however, whether or not these compounds or the cycle reactions play essential roles in the metabolism of the mold. It is to be noted that the mutants which cannot carry out reactions within the cycle (10575 and C83) grow poorly, as compared to all the other mutants of the series, even with an excess of tryptophane. This fact suggests a special metabolic significance to turnover in the cycle.

Summary.—1. Tryptophane is shown to disappear rapidly from culture medium which is inoculated with any one of the Neurospora strains tested. One of the products of tryptophane breakdown has been isolated and identified as anthranilic acid.

2. Evidence is presented which suggests that in Neurospora tryptophane, anthranilic acid and several other compounds are involved in a metabolic cycle.

3. The evidence demonstrates that 3-hydroxykynurenine is an intermediate in tryptophane metabolism in Neurospora, but indicates that it is not directly involved in the cycle.

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THE CORRELATION IN SHAPE AND SIZE BETWEEN EPIDERMAL AND SUBEPIDERMAL CELLS

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In these PROCEEDINGS in 1945, Matzke[1] published his remarkable tabulation showing that in a foam of bubbles of equal volume 400 of those at the periphery had an average of 10.99 facets. An average close to 11 was not unexpected, but the precision of his result is impressive. The general shape which such bubbles, or comparable simple vegetable cells, could take uniformly has recently been figured by the writer,[2] using a conventional model. Each peripheral cell has a free surface which is hexagonal; six lateral surfaces, four of which are pentagons, and the other two, on opposite sides, are quadrilateral; and four basal facets, two quadrilateral and two hexagonal. It is immediately apparent that this shape does not depend on uniformity in volume with the cells beneath, since all the conditions may be met by tall prismatic 11-hedral cells of relatively large volume, or by flat hexagonal plates like a thin tiling. But the area of the peripheral or epidermal cell in a tangential section of a cylindrical stem should be the same as that of the underlying or subepidermal cell. Then this pattern may be

recognized though, through cell division, the 11-hedron has become an average of diverse shapes and sizes, or, as in bubbles, it is imperfectly presented through casual irregularities.

Frequently the epidermal cells are smaller, or sometimes larger, in tangential section than the cells immediately beneath. This paper deals with the effect of such differences upon the typical pattern. The problem was first encountered by the writer in 1928,[3] in studying the epidermis of the cucumber, and was tentatively solved by use of the diagram here reproduced as figure 1. When the average hexagonal cell of the epidermis has the same transverse dimensions as the cells below, the interrelation of the bases of the epidermal cells and the tops of the underlying cells appears as in *A*. The subepidermal cells are lightly outlined in the figure. It may be as-

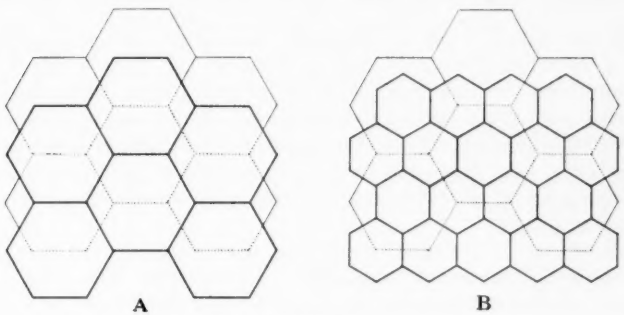

FIGURE 1

A.—Layer of regular hexagons superimposed on a layer of regular hexagons of the same area so that the resulting pattern is a projection of tetrakaidecahedra. *B*.—Layer of regular hexagons superimposed on regular hexagons of three times their area. The underlying cells become 17-hedra: the overlying cells are one 8-hedron for two 10-hedra. (*Anat. Rec.*, **38**, 352 (1928)).

sumed, as is often the case, that cells of the *second* layer beneath the epidermis are of the same size as those in the first, so that the subepidermal layer rests upon one which duplicates it. Then the subepidermal cells, if uniform, are 14-hedra, hexagonal in cross-section. They have 6 lateral facets and end in 4 facets above and below. When a cell of this size and description becomes an epidermal cell, a single facet—the hexagonal free surface—replaces one of its 4-faceted ends and makes it 11-hedral. Should the upper and the lower ends each be replaced by a single free surface, the 14-hedron is reduced to an 8-hedron, hexagonal above and below, with its 6 lateral surfaces all quadrilateral.

An elegant demonstration of the 8-hedral form of cells with free surfaces above and below has recently been provided by Miss Flint[4] in her study of cell shape in the aquatic

plant *Utricularia inflata*. Its floats (Schwimmorgane), as figured by Goebel, are bladder-like formations containing an axial strand from which 6 plates of cells, one cell thick, radiate to an enclosing epidermis, also one cell thick, thus dividing the bladder into air-filled compartments. Though the outlines of the septal cells on surface view are wavy, suggesting a jig-saw puzzle, whereas those of the epidermis are "regular," they both present the common range for polygonal cell shapes, from quadrilaterals to nonagons, inclusive, and their average number of sides is nearly 6. If one accepts the geometrical conclusion that, apart from vertices of 4 or more rays and the requisite deduction of 12 sides for a mosaic that completely encloses a space, the average number of sides for such cells must be precisely 6, then the averages recorded are due to the chance of selection. For the free surfaces of 200 septal cells Miss Flint found an average of 5.875 sides, and for 200 epidermal cells 5.825. Every cell had a corresponding number of lateral quadrilateral facets, averaging therefore, instead of 8.0, 7.875 and 7.825 facets, respectively. Large air spaces in the midribs of certain leaves offered a third source of such cells; 200 of them had free surfaces averaging 5.905 sides and a corresponding average of 7.905 facets.

In the cucumber the subepidermal cells, from a few measurements of their transverse diameters, were estimated to have cross-sectional areas about three times as large as the epidermal. Subsequent planimeter measurement of a patch of 100 epidermal cells in that specimen, and of 100 related subepidermal cells in large groups, showed their relative areas more closely as 1:2.66. Since the difference in size develops gradually, there is no expectation that the actual ratios should be the simple fractions used in this study.

Showing precisely the 1:3 proportion, the diagram figure 1, *B*, was presented as "a theoretical abstraction." Its epidermal cells occur in two forms—10-hedra, with 3-faceted bases, surrounding 8-hedra with single-faceted bases, placed over 17-hedral subepidermal cells. Although central 8-hedral epidermal cells certainly occur in this epithelium, they are not sufficiently numerous to form one-third of all its cells as in the pattern figured. This arrangement would reduce the average number of epidermal facets from 11 to $9\ ^1/_3$.

A further criticism of the diagram is that the orientation of the hexagons is prosenchymal in one layer and parenchymal in the other. Since these orientations depend on the predominant direction of growth, in girth or in length, respectively, they usually accord in adjacent layers. In the cucumber, expanding comparably in both directions, the orientation is similar, though confused, in these two strata.

When a network of regular hexagons is placed over a network of larger regular hexagons in the same orientation, a complex pattern results which is far more suggestive of the cucumber tissue. By making 7 apothems of the small hexagons equal to 4 apothems of the large, the area of the small will be nearly one-third (32.65 per cent) of the large, and a pattern formed by 49 small hexagons over 16 large ones may be repeated indefinitely. Seven of the 49 small hexagons will provide 8-hedral central or enclosed cells with

single-faceted bases (like those in Fig. 1, *B*) and two more will be added after the elimination of coincident edges. The average number of facets for the epidermal polyhedra becomes 9.47, and for the subepidermal, 17.56.

The next tissue to be studied in this respect was fortunately the simpler epidermis of the stem of *Tradescantia*.[5] Its hexagonal cells so far approach rectangular forms that they may be treated as such while recognizing the full significance of certain rounded corners and edges. The epidermal and subepidermal cells, primarily of similar dimensions, increase equally in width, and would also in length except that transverse divisions are more frequent in the epidermis. Hence, though the tissue comes to consist of cells elongated lengthwise of the stem, those of the epidermis are only one-

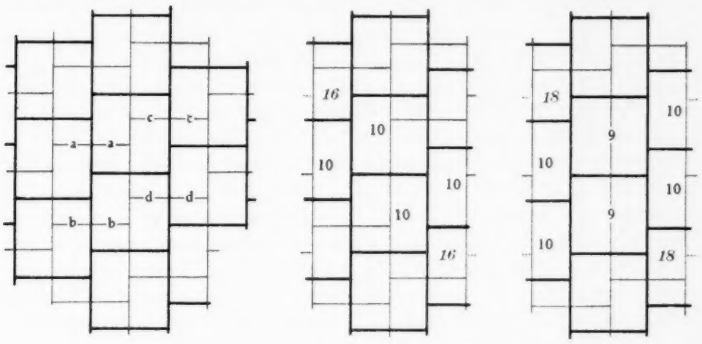

FIGURE 2

A.—Rectangular hexagons of the same size superimposed as in Fig. 1, *A*. *B*.—Middle portion of *A* with septa *aa* and *dd* removed. Roman figures indicate the facets of the upper "cells" (always 7 plus their basal facets): italics are the ultimate facet numbers for the cells below (always 10 plus their top facets). *C*.—Like *B*, but with 4 septa removed.

third of the length and area of the subepidermal. An average of $9\ ^2/_3$ facets for the mature epidermal cells has been reported, and of 18 facets for those immediately beneath. This contrasts with the epidermal average of $9\ ^1/_3$ and subepidermal of 17 facets originally assigned to the cucumber, and also with the revision of $9\ ^1/_2$ and $17\ ^1/_2$ respectively, now adopted to provide for similar orientation. The significance of these differences may be explained as follows.

When the epidermal and subepidermal cells have the same area in cross-section and occur in rectangular form, their mutual relations may be shown in the diagram figure 2, *A*, essentially like figure 1, *A*. Its heavily outlined epidermal cells are all 11-hedral, and its lightly outlined subepidermal are all 14-hedral. The removal of septum *aa*, or any one like it, makes a sub-

epidermal cell of twice the area of the epidermal, whereupon two of the latter each lose a basal facet and become 10-hedral. The removal of another septum (*dd*) may produce two more 10-hedra (Fig. 2, *B*), or, if septum *cc* is the second to go, one 9-hedron and two 10-hedra with the same combined loss in epidermal facets. A general reduction of the epidermal cells to half the cross-sectional area of the subepidermal makes the average epidermal cell 10-hedral, and the subepidermal 16-hedral. The latter then have 4 basal facets (when the cells beneath them have the same dimensions), 6 lateral facets, and 6 toward the epidermis (Fig. 2, *B*).

The removal of septa *aa* and *bb* in figure 2, *A*, forms an underlying cell of three times the area of the related epidermal cells, four of which have become 10-hedra; or, as the process extends by taking out *cc* and *dd*, the condition in figure 2, *C*, results, essentially like that observed in *Tradescantia*. The epidermal cells one-third the size of those beneath, and presenting two 10-hedra for one 9-hedron, have an average of $9\ ^2/_3$ facets. The subepidermal cells, 8-faceted toward the epidermis, acquire an average of 18 facets.

A remarkable feature of this transformation, effected in cells by the introduction of epidermal septa and not by the removal of subepidermal, is its continued accordance with the simple formula established when the epidermal and subepidermal cells have the same average cross-sectional areas (as in Figs. 1, *A* and 2, *A*). Then, unless a space is completely covered or enclosed, if *np* is the number of peripheral or epidermal cells, and *nc* the number of central or subepidermal cells, their total number of facets, *F* equals $11np$ plus $14nc$. In figure 2, *C*, for example, the pattern reduces to a multiple of 3 peripheral for 1 central, having $29 + 18$ or 47 facets, whereas by the formula the same total is found as $33 + 14$.

If the situation in the diagram figure 2, *C*, is considered reversed, with the lightly outlined cells as epidermal and the smaller dark ones as internal and subtended by a layer of their own size, for one epidermal with 15 facets there would be two subepidermal with 13 and one with 12 facets, 53 facets altogether. By the formula there should be $11 + (3 \times 14) = 53$.

The cucumber, however, with the same relative cell areas as *Tradescantia* which complies with the formula, affords a definite deviation. Instead of one 9-hedron for two 10-hedra as in figure 2, *C*, the cucumber, as interpreted in figure 1, *B*, would have one 8-hedron for two 10-hedra; and in place of an average subepidermal 18-hedron it would substitute a 17-hedron. These differences are associated with the presence of enclosed cells with 1-faceted bases, not found in the patterns of figure 2. In vertical sections of the epidermis this difference is strikingly evident, since with cells showing the ordinary 2-sided bases, the cucumber has 36% of them 1-sided and 2% 3-sided, whereas in *Tradescantia* 1% are 1-sided and 24% 3-sided. These come and go with growth and cell division in a shifting pattern.[5]

The purely diagrammatic figure 3, in which the epidermal cells of one-

third the sectional area of the subepidermal are shown in the rectangular form of *Tradescantia* while presenting the deviation in the cucumber, demonstrates the effect of the single basal facet. Throughout the pattern for one 8-hedral epidermal cell with a 1-faceted base, there are two 9-hedral, two 10-hedral and one 11-hedral—6 epidermal cells with 57 facets, over 2 subepidermal with 35 facets—92 altogether, where the formula calls for $66 + 28 = 94$. There is a loss of two facets, one in the epidermis and the other, its duplicate, in the subepidermis, for every cell with a single-faceted base.

This conclusion rests only on the several instances studied. In the superimposed networks of regular hexagons already cited, the smaller having areas approximately one-third of the larger, 49 "epidermal" bases are over 16 "subepidermal" tops. Whereas the corresponding cells by the formula $11np + 14nc$ should have 763 facets, the geometrical pattern, which has 9 single-faceted bases, has 745 facets, a deduction of 2 for each of them. By making 7 apothems of the small hexagons equal to 5 of the large, the small ones are about half the size (51%) of the others. When superimposed, 49 epidermal with 4 single-faceted bases are over 25 subepidermal. By the formula there should be 889 facets, and there are 881.

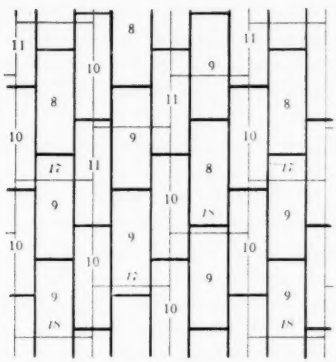

FIGURE 3

Layer of rectangular hexagons over a layer of others three times as large. The facet numbers are as in figure 2, *B*.

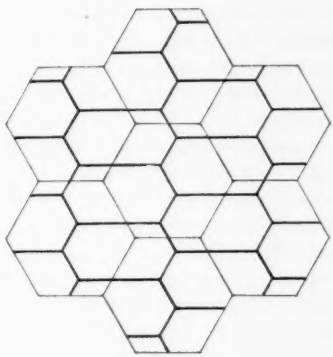

FIGURE 4

Diagram of 10-hedral epidermal cells over 16-hedral subepidermal cells of twice their area.

The extent to which cells with single-faceted bases occur in the vegetable tissues studied statistically has not been determined. Obviously impossible with cells of the same area as those beneath, and difficult to produce in those of nearly the same size, it can occur with great frequency when the areas are as 1 to 3. In a layer of small cells between two layers of large ones, arranged as in figure 1, *B*, the average facets of the small internal cells could be reduced from 14 to $10\ ^2/_3$, and there are many possible intermediate reductions. Perhaps this range includes "the internal cells of

the young midrib" of *Utricularia*, 100 of which average 12.34 facets per cell as studied by Miss Flint.[4]

A nearly perfect example of the epidermal geometry has been found by Matzke[6] near the stem tip of *Elodea*, where the "subepidermal cells were somewhat larger than those of the epidermis." There he found that 200 epidermal cells had an average of 10.005 facets, and that their most frequent shape, presented by 34 of them, was the 4-4-2 pattern (4 quadrilateral facets, 4 pentagonal and 2 hexagonal). In our diagram (Fig. 4) the underlying cells are twice the area of the epidermal, and all of the latter are 10-hedra of the 4-4-2 pattern. The free surfaces of the epidermal cells have the shape that Goebel drew in the little seaweed *Giraudia*,[7] which so pleased Thompson that he modified the picture and placed it in *Growth and Form* with a confusing misspelling as the "Epidermis of *Girardia*."[8] Figure 4, which is a variety of figure 2, *B*, is not presented as a picture of *Elodea*, or exactly of any other plant, but as a diagram which may help to put in order the observed confusion. The subepidermal hexagonal mosaic which presents 2 vertices (or an average of 2) to be covered by each epidermal cell of the same size, giving it a 4-faceted base, presents a single vertex yielding a 3-faceted base for epidermal cells of half the size and double the number of the subepidermal. Additional epidermal cells, finding no vertex available, may form either 2-faceted bases over an edge (*Tradescantia*), or single-faceted bases over the surface away from the corners and edges (cucumber).

Summary.—Because of the geometric control of cell shapes in plants, the previously reported average of 9 $^1/_3$ facets per epidermal cell resting on subepidermal cells of three times their area, in the cucumber, and 9 $^2/_3$ facets per epidermal cell resting on subepidermal cells of three times their area, in *Tradescantia*, appeared as a discrepancy in need of explanation. The 9 $^2/_3$ average, which accords with the geometrical formula whereby epidermal cells primarily average 11 facets and internal cells 14 facets, has been verified. The epidermal cells of the cucumber, revised as having an average of 9 $^1/_2$ facets, remain a deviation which depends on the occurrence of epidermal cells with single-faceted bases. For epidermal cells one-third the area of the cells beneath, the presence of those with single-faceted bases in the proportion of 1 in 6 would account for a reduction in the average epidermal facets from 9 $^2/_3$ to 9 $^1/_2$.

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THE EFFECT OF NITROGEN MUSTARD PRETREATMENT ON THE ULTRA-VIOLET-INDUCED MORPHOLOGICAL AND BIOCHEMICAL MUTATION RATE[1]

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A wide variety of experimental studies have indicated that the stability of genes to mutagenic radiations resides, to a large extent, in the inherent nature of the individual genes, and is independent of extrinsic factors to which the genes can be experimentally exposed. While it is known that the same genes may respond differentially to x-rays when in different tissues, e.g., spermatogonia as opposed to mature spermatozoa,[3] the rather constant rate of mutability of individual genes, and even of alleles of the same gene, is somewhat surprising since a rather diverse group of supplementary treatments have been shown to influence the rate of detectable x-ray-induced chromosomal rearrangements, and since it also is known that for this type of radiation there exists a close, if unresolved, relationship between rearrangements and mutations.[4] Even when the x-ray-induced frequency of mutations can be increased by non-mutagenic supplementary treatments such as low temperatures,[5] infra-red[6] and by high versus low intensities,[7] or decreased by exposure *in vacuo*,[8] the conclusion that these data reveal a dependence of genic stability upon environmental factors is open to suspicion. Such induced mutations and, in particular, the recessive lethals as found in Drosophila, do not form a homogeneous group, some being associated with chromosome breakage, others ostensibly not, and some being reversible, others not.

The mutations induced by ultra-violet radiation are less suspect of being associated with chromosomal rearrangements, ultra-violet being incapable of inducing the variety of intra- and inter-chromosomal changes characteristically produced by x-radiation.[9, 10] Therefore, when it was found[11] that a 30-minute pretreatment of the spores of *Aspergillus terreus* with a non-mutagenic aqueous concentration (0.1%) of nitrogen mustard greatly increased the mutability of the genes to ultra-violet (2537 *A*), it appeared that the technique offered an opportunity to investigate further the possibility of materially altering the stability of the genes to subsequent exposure to ultra-violet. In the above experiment, only morphological mutations were considered, and since *A. terreus* possesses no known sexual stage, the genetic analysis could be carried no further. Consequently, some doubt remained concerning the classification of morphological mutation as true point mutations, and a more refined approach to the question of altered mutability was needed. Additional experiments were therefore carried out

on a microconidial strain of *Neurospora crassa*.[12] The frequency of both morphological and biochemical mutations was determined, but rather than test the morphological mutations for their transmissibility and possible associated sterility by means of the ascospore-isolation technique, the biochemical mutations because of their homogeneity were considered to be as close an approximation to true point mutations as was possible. Most of them can be readily mapped,[13] and many of them are reversible in nature.[14] There is also less likelihood that terminal deletions, which are the type of chromosomal change predominantly induced by ultra-violet, would lead to viable biochemical mutations without associated morphological change in a haploid organism such as Neurospora.

The technique of treating the spores with ultra-violet has been described.

TABLE 1

FREQUENCY OF ULTRA-VIOLET-INDUCED MORPHOLOGICAL MUTATIONS IN *Neurospora crassa* WITH AND WITHOUT PRETREATMENT WITH NITROGEN MUSTARD

TREATMENT, IN MINUTES	NO. ISOLATES	NO. MUTANTS	MUTATION, %	P VALUE
Control	1018	4	0.39	0.15
Mustard (0.1%)	1308	12	0.92	
UV—2*	588	24	4.1	<0.0001
M + UV—2	541	72	13.3	
UV—5	1982	140	7.1	<0.0001
M + UV—5	1858	208	11.2	
UV—7½	623	118	18.9	0.002
M + UV—7½	725	202	27.9	
UV—10	1698	290	17.1	<0.0001
M + UV—10	1358	358	26.2	
UV—12½	2190	340	15.5	
M + UV—12½	1640	254	15.8	

* UV given at a rate of approx. 130 ergs/cm.²/sec.

Eight-day old cultures were used throughout the experiment. The mustard employed was bis-beta-chloroethylmethylamine(HCl), and the 30-minute pretreatment consisted simply of suspending the microconidia in a 0.1% aqueous concentration. The spores were washed before radiation, irradiated and then plated onto a complete medium to which 3% *l*-sorbose had been added.[15] The isolates were transferred to a complete medium, scored for morphological mutations and then transferred to a liquid minimal medium for the detection of biochemical mutations. Only those isolates which exhibited the growth characteristics of the parental strain, but which did not grow on a minimal medium, were classified as possessing a biochemical mutation. Most of these could be shown to involve single amino acid or vitamin deficiencies, although a few grew only when a complex substrate

was provided. Not all of the isolates were tested for biochemical mutations (compare the number of isolates in tables 1 and 2), and some of the morphological mutants were untestable biochemically because of their non-sporulating nature and thin mycelial growth.

Table 1 summarizes the data dealing with the frequency of morphological mutations. These were recognized by their effect on the growth characteristics of the colonies on a complete medium. Very few color mutants were observed because of the albinistic nature of the strain. There was no increase in the frequency of morphological mutations when transfer was made to the minimal medium, indicating that this type of mutant was not dependent upon any simple biochemical requirement.

Unlike *A. terreus*, this microconidial strain was sufficiently sensitive to the 0.1% concentration of nitrogen mustard to yield a percentage of mutations more than twice that of the untreated controls, an increase, however, which was not significant at the 1% level. Except for the $12^1/_2$-minute series, the mustard-ultra-violet treatments showed a highly significant in-

TABLE 2

FREQUENCY OF ULTRA-VIOLET-INDUCED BIOCHEMICAL MUTATIONS IN *Neurospora crassa* WITH AND WITHOUT PRETREATMENT WITH NITROGEN MUSTARD

TREATMENT, IN MINUTES	NO. ISOLATES	NO. MUTANTS	MUTATIONS, %
Control	750	0	0.00
Mustard (0.1%)	1003	0	0.00
UV—2	587	2	0.34
M + UV—2	531	3	0.56
UV—5	902	4	0.44
M + UV—5	825	6	0.73
UV—$7^1/_2$	603	6	1.00
M + UV—$7^1/_2$	701	12	1.70
UV—10	1698	17	1.01
M + UV—10	1358	23	1.69
UV—$12^1/_2$	1406	14	1.00
M + UV—$12^1/_2$	724	15	2.07

$X^2 = 20.1913$ $N = 4$ $P < 0.01$

crease in frequency of mutations over that obtained from the ultra-violet controls. These data are in substantial agreement with similar data obtained from *A. terreus*,[11] where the mustard pretreatments gave high increases at the low ultra-violet dosages, raised the peak of the mutation curve and caused a more abrupt decline in frequency at the higher dosages. The microconidia, however, were so much more sensitive to ultra-violet than were the spores of *A. terreus* that it was not possible to obtain large numbers of isolates at the higher dosage levels, with the consequence that the rate of decline in frequency for the two series could not be compared.

Table 2 summarizes the data dealing with the frequency of biochemical mutations. Since no comparable data of this sort have been previously published, figure 1 is included for graphic purposes. Neither the untreated nor the mustard control series yielded any mutations of this nature. A consistently higher frequency was obtained from all of the mustard-ultra-violet treatments as compared with their respective ultra-violet controls, but unlike the downward trend found in the rate of morphological mutations no fall in frequency of biochemical mutations was evident. The mustard-ultra-violet rate continued to rise throughout the range of dosages employed, while the ultra-violet controls reached a peak at the $7^1/_2$-minute dosage, and continued at the same level through the 10- and $12^1/_2$-minute intervals. Whether an eventual decline in frequency would take place at

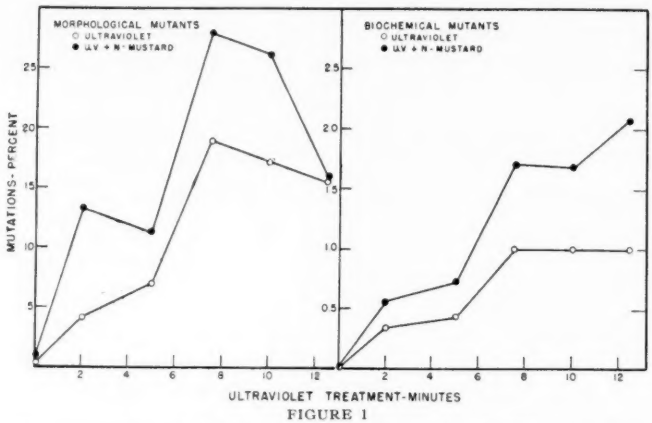

FIGURE 1

Frequency of morphological and biochemical mutations in *Neurospora crassa* as a function of ultra-violet dosage.

still higher dosages could not be determined because of the very low survival rate of the microconidia.

Of the five ultra-violet dosages recorded in table 2, only the $12^1/_2$-minute series showed a highly significant increase with the mustard pretreatment, with the 10-minute series being significant at the 5% level only. The remaining three dosages gave no individually significant increases, but the summated chi-square values for the five series, as computed by means of a five-place contingency table, yielded a P value of <0.01.

It appears reasonably certain, therefore, that the frequency of both morphological and biochemical mutations can be materially altered by pretreatment with a non-mutagenic concentration of nitrogen mustard. It

is not possible to state what the nature of the sensitization mechanism might be, but there can be little doubt that the frequency of mutation under these experimental conditions is not solely a function of the amount of radiant energy absorbed. Whether the altered frequency has resulted from an altered stability of the genes, or from an altered frequency of extragenic changes which have a phenotypic expression (e.g., position effects), could not be determined with absolute certainty, but the data on biochemical mutations, which represent a relatively homogeneous group, and the known action of ultra-violet on genes and chromosomes, suggest that the former alternative is more likely the correct one.

The data on the frequency of biochemical mutations as a function of dosage also have a bearing on the interpretation of the shape of the morphological mutation curve. It has been proposed that the decline in frequency of morphological mutations at the higher dosages of ultra-violet results from a selective killing action which favors the non-mutated spores, i.e., the deleterious action of the ultra-violet on the cytoplasm makes more difficult the germination and growth of the mutated spores.[6, 16, 17] On this hypothesis, the biochemical mutations, being essentially wild type when grown on a complete medium, should possess no selective disadvantage, and should continue to increase as the dosage is increased. Support for this is provided by the data in table 2 where no fall in frequency of biochemical mutations was found at dosages where the morphological frequency was declining, although a leveling off of mutation rate is found for the ultra-violet control series.

Summary.—Pretreatment of the microconidia of *Neurospora crassa* with a relatively non-mutagenic concentration of nitrogen mustard alters the stability of the genes in such a manner as to increase significantly the effectiveness of ultra-violet in inducing morphological and biochemical mutations. It is suggested that the increased frequency of mutations results from an increased frequency of intra-genic rather than extra-genic (chromosomal) alterations.

[1] Supported by grants from the National Institutes of Health, U. S. Public Health Service.

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*THE PRINCIPLES OF TIME SERIES ANALYSES APPLIED TO OCEAN WAVE DATA**

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Introduction.—In the original report on the auto correlation analysis of ocean wave data the apparent advantages of that method over periodogram and Fourier series applications to finite amounts of data were brought out.[1] Because of practical restrictions in that only finite amounts of observational data are available for study, the fitting of wave heights as functions of time by means of trigonometric components is only in special circumstances suitable for describing the physical characteristics of the data itself. In the case of ocean wave data, *a priori* knowledge of frequency is not available, and results from this type of analysis have resulted in claims that sea surface roughness patterns are composed of large numbers of frequencies which have traveled with independent velocities from their regions of generation. In other words, mathematical representations of the data have been assumed to be physically significant, a situation which may lead to apparent misconceptions of the behavior of the phenomenon and of the dynamics causing it. In fact, the fitting of data by any of the various methods of cycle analyses, in the absence of *a priori* knowledge of the physical properties, is no more effective than the fitting of that same data, say, by orthogonal polynomials.[2]

It appears that the problem of understanding the behavior of a physical phenomenon which varies with time is directly associated with that of pre-

dicting its future course from its past behavior, and the distance into the future which the phenomenon may be accurately forecasted may be taken as an indication of the amount of dynamics associated with the phenomenon itself. Thus, in the case of sea surface wave data, while a mathematical theory may be set up from the results of a periodogram or Fourier series analysis to explain a record, it has little physical meaning unless it can be extrapolated into the future. It was from this point of view that Wiener[3] investigated the problem of prediction and defined the dynamics of a time series in terms of the possibility of prediction into the future. The auto correlation of a time series provides a means of doing this. It can be shown that the auto correlation function may be obtained from the spectrum of the series, and the spectrum may also be obtained from the auto correlation function. In this process the spectrum and the auto correlation function of a discrete sequence of observations preserve information by retaining the frequency and the amplitudes of all frequencies from zero to infinity, whereas, on the other hand, it discards information relative to phase relationships.

TABLE 1

RECORD NO.	LOCATION	DEPTH, FT.	DATE	STARTING TIME	RECORD LENGTH, SECS.
B-713a	Bermuda	120	2/21/47	1840	350
B-12a	Bermuda	120	10/25/46	1405	350
1–5B	Cuttyhunk	75	6/14/46	1149	60
39-L	Cuttyhunk	75	7/ 4/46	0627	350
53-W	Cuttyhunk	75	9/15/46	0050	350
53-X	Cuttyhunk	75	9/15/46	0650	350
53-Y	Cuttyhunk	75	9/15/46	1250	350
56-E	Cuttyhunk	75	10/ 1/46	1244	350

The method of auto correlation analysis is employed in the analysis of many types of scientific problems. In addition to its use as a prediction mechanism, the correlograms and the Fourier transforms of the correlograms into power spectra serve to describe the basic data by measuring its physical properties of period, amplitude and damping coefficients which, in turn, may be employed in an attempt to discover something of the physical causes. The results throw new light on the physical properties of the sea surface and on the behavior of ocean wave patterns. The method provides a means for distinguishing between mathematical and realistic physical representations of sea surface roughness patterns.

Computation of the Auto Correlation Coefficient.—Theoretical formulae defining the auto correlation coefficient as developed by Wiener,[3] Khinchine[4] and others are not strictly applicable to practical computations.[5] This is because wave heights are not observed as infinite functions of time (very frequently only short series of observations are obtainable), and also be-

cause the means of such finite series frequently depart from zero. In our computations it was found necessary to use a running mean rather than extracting a grand mean at the very beginning of the computation. To compute the auto correlation coefficient we compute the following statistics from N equally spaced observations for k equally spaced time lags.

$$T_0 = \sum_{i=1}^{N} x_i, \qquad T_k = \sum_{i=1}^{N-k} x_i, \qquad T_k' = \sum_{i=1}^{N-k} x_{i+k}$$

$$S_0 = \sum_{i=1}^{N} x_i^2, \qquad S_k = \sum_{i=1}^{N-k} x_i x_{i+k}$$

$$S_0 = \frac{NS_0 - T_0^2}{N^2}, \qquad S_k = \frac{(N-k)S_k - T_k T_k'}{(N-k)^2}$$

$$r_k = S_k/S_0$$

The correlogram is obtained by plotting the auto correlation coefficient (r_k) as ordinate against k as abscissa. In this particular investigation the correlogram provides the basis for analysis of the data, and may be transformed into a power spectrum by a Fourier transform of the auto correlation coefficients. This spectrum is identical with that which would be obtained by periodogram analysis of an infinite series.

In the applications considered in this paper the auto correlation function frequently occurs as a composite of a pure sinusoidal wave and one producing a continuous spectrum. The period of the former is given by the correlogram and its amplitude computed from the original data by least squares. The residue after subtraction from the original data consists of a continuous spectrum. This provides a practical example of theorem 3 of Khinchine's theory of correlation[4] in which the auto correlation function is split into a line and a continuous spectrum.

The Data.—This paper presents results of auto correlation analysis of eight ocean wave records. The records are from underwater pressure recorders located at Cuttyhunk, Massachusetts (depth 75 feet), and at Bermuda (depth 120 feet) and represent pressure pulses at the sea bottom transmitted from the sea surface.[6] For the analyses, the records were scaled for pressure values at intervals of one second and were not reduced to sea surface wave heights.[7]

Bermuda Record B-713a is discussed separately. Table 1 summarizes the records analyzed.

The Analysis of Wave Record B-713a.—This record is discussed in more detail than the others for comparison of the results from cycle analyses with those of auto correlation analyses on finite amounts of data. Cycle analysis refers both to mechanical periodogram analysis and to the fitting of data by sums of sines and cosines. Results obtained by the former method, to which the data had already been subjected,[8] are first discussed in the

light of their usual interpretation. This is then supplemented by computation of certain statistical properties of the data and then followed by fitting a series of cosine waves to the data over the critical part of the spectrum. Results obtained by the above methods are then compared with those obtained by the method of auto correlation analysis.

a. Mechanical Periodogram Analysis: The mechanical periodogram analysis of this 20-minute wave record (Fig. 1) shows a continuous spectral distribution with about 32 defined peaks between 8.8 and 40 seconds. The greatest development occurs in the 8- to 10-second band, and maximum energy is indicated for 8.8, 9.0 and 9.5 seconds. The customary interpretation given this type of analysis is that the basic wave pattern is made up of

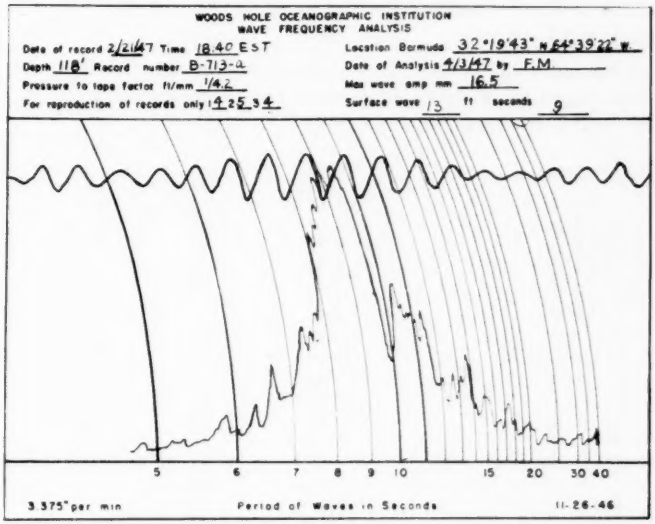


FIGURE 1

Results of mechanical periodogram analysis of Wave Record B-713a. See text.

the frequencies indicated. The principal wave periods lie between 6 and 15 or 20 seconds, with the maximum period band of the "wave train" extending from 8 to 10 seconds. The characteristics of different wave records are usually compared by the widths and locations of the maximum band development, together with locations of the individual maximum points. Special consideration is frequently given to indications of longer periods, say above 20 seconds, which are sometimes presumed to be waves generated by oceanic storms at great distances but which have "out-run" the storm itself.

The above physical interpretation is doubtful in that periodogram analyses of finite amounts of data are not necessarily physical, but may be only mathematical representations of the data. Its acceptance as a physical representation without *a priori* reason may lead to a misconception of both the physical properties and the causative mechanism of the sea surface roughness. This question is discussed later after additional results are developed.

b. Statistical Characteristics: The frequency distribution of periods, compiled from measurements of the times between successive wave crests on the record, has the following properties:

Standard deviation = 1.135 seconds
Mean = 8.59 ± 0.077 second
Mode = 8.75 seconds

Thus, the mean and modal values of distances between wave crests in the original record lie in the range of maximum energy brought out by the mechanical periodogram analysis (Fig. 1).

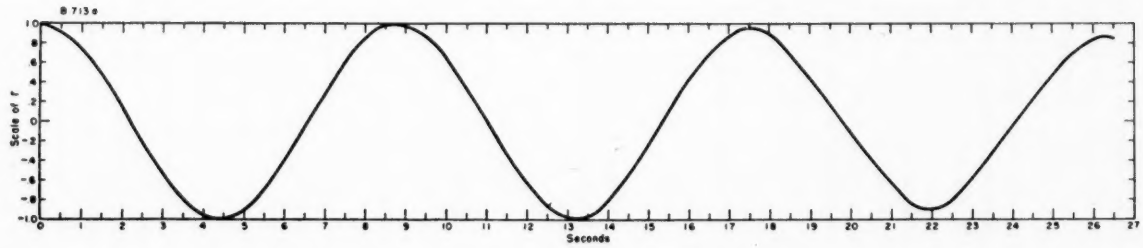
TABLE 2

PERIOD	RELATIVE AMPLITUDE	PHASE ANGLE, °
7.50	1.00	156.6
8.00	2.31	111.9
8.25	2.97	23.6
8.50	4.01	30.1
8.75	4.60	18.9
9.00	5.29	69.6
9.25	10.93	276.9
9.50	3.49	231.2
9.75	2.20	210.2
10.00	0.77	16.9

Amplitude and phase values obtained by fitting cosine waves to Wave Record B-713a.

c. Cosine Wave Analysis: Results obtained by independent least square fits of ten cosine waves (periods from 7.5 to 10.0 seconds) to the basic data are tabulated in Table 2. The spectrum is similar to that obtained by mechanical periodogram analysis, increasing from a minimum at 7.5 seconds to a maximum value at 9.25 seconds and then diminishing rapidly to a minimum at 10.0 seconds. This representation, obtained from a finite amount of data, is like that of the mechanical periodogram analysis—of mathematical significance only and without physical reality. The rôle of this type of analysis is a secondary one, to be applied after the period, or periods, have been established by other means.

d. Auto Correlation Analysis: The correlogram of the primary data of Wave Record B-713a (Fig. 2) maintains a period of 8.75 seconds and damps



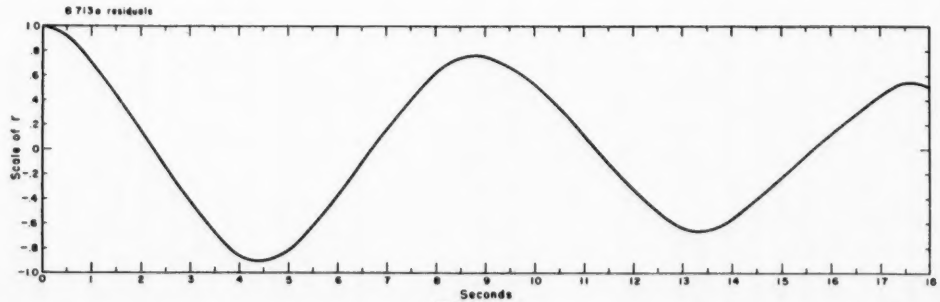


FIGURE 2

Correlograms of Wave Record B-713a. Curves have been drawn through values of r computed for each second. The curves have not been smoothed.

slowly; at the end of the third cycle the amplitude is reduced to a value of $r = 0.87$, after which it continues to damp slowly until reaching a terminal value (r_T). The terminal amplitude is maintained by a cyclical component which proceeds undamped through the basic data. It is the fundamental wave to which energy is supplied by an external force at a rate equal to that at which it is dissipated in the system. Its amplitude, obtained from a least square fit of an 8.75-second sine wave to the primary data, is 0.720 foot.

Subtracting the cyclical component ($T = 8.75$ seconds, $A = 0.720$ foot) from the original data, the variance reduces from a value of 0.492 square foot to 0.216 square foot for the residuals. Recomputing the auto correlations of the residuals, the correlogram then damps in a fashion characteristic of an autoregressive series; at the end of the second cycle, the correlogram amplitude is $r = 0.54$ as compared to $r = 0.95$ for that of the primary data (Fig. 2).

The value of the terminal amplitude of the original correlogram is

$$r_T = \frac{\text{Variance cosine}}{\text{Total variance}} = \frac{A^2/2}{\sigma_y^2} = 0.56$$

for Wave Record B-713a. This value also represents the per cent reduction due to the cyclical component itself, and may be expressed as the ratio of the residual (σ_r^2) and original (σ_y^2) variances. Thus:

$$r_T = 1 - \frac{\sigma_r^2}{\sigma_y^2}$$

The numerical solutions of the above and other relations, for cases where the original series consists of a sine term and a random residual, provides a means of checking the computations.

The significant difference between the results of periodogram analyses and correlogram analyses of finite amounts of data is that the former are not the power spectra of the latter. The large number of peaks in the periodogram of Record B-713a (Fig. 1) are not real periods, and they provide only a mathematical representation of the data. The spectrum is similar to that obtained by the arbitrary fitting of cosine waves to the data, and is not to be interpreted as having physical reality.

On the other hand, the use of auto correlation analysis reveals that for Record B-713a the auto correlation function is broken into two parts—one having a line, and the other a continuous spectrum. The line spectrum is that of the cyclical component present in the basic data, whereas the continuous spectrum represents a wide band of frequencies, with an average frequency equal to that of the cyclical components, but which damp out rather rapidly.

The physical interpretation to be given results from the two types of

analyses is equally significantly different. The auto correlation analysis suggests that the physical characteristics of the sea surface wave pattern comprise one fundamental wave with a period of 8.75 seconds and an amplitude of 0.72 foot, on which is superimposed a series of local impulses. The former, generated presumably by a meteorological situation acting at some distance from the target, is the predictable component in the data. It accounts for an estimated 56% of the total variability of the sea surface pattern (obtained from the value of the terminal amplitude of the correlogram) and lends itself to being forecast from knowledge of wind action on the sea surface. The remainder of the data represents a superimposed

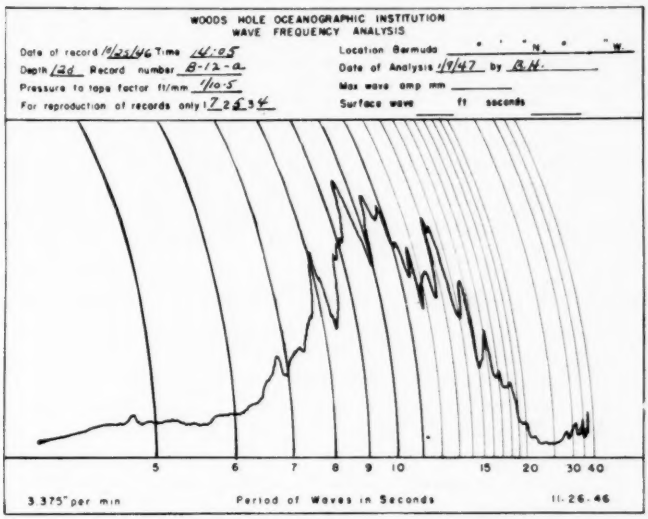

WOODS HOLE OCEANOGRAPHIC INSTITUTION
WAVE FREQUENCY ANALYSIS
Date of record 11/22/46 Time 14:05
Depth 12d Record number B-12-a
Pressure to tape factor ft/mm 1/10·5
For reproduction of records only 1 2 5 3 4
Location Bermuda
Date of Analysis 1/9/47 by B.H.
Max wave amp mm
Surface wave ft seconds
5 6 7 8 9 10 15 20 30 40
3.375" per min
Period of Waves in Seconds
11-26-46

FIGURE 3

Results of mechanical periodogram analysis of Wave Record B-12a. See text.

series of local impulses which die out rather rapidly. They would appear to be generated by local winds in the area of the target, and their relative significance to the pattern is also evaluated from the terminal amplitude of the original correlogram.

Auto Correlation Analyses of Additional Ocean Wave Records.—The results of auto correlation analyses of seven additional ocean wave records from Cuttyhunk and Bermuda are tabulated in Table 3. The mechanical periodogram analyses of these records gave results similar to those for Record B-713a, each showing approximately 30 peaks. Figures 1 and 3 are representative of this analysis.

The original correlograms do not damp completely (Figs. 2 and 4),[9] terminal amplitudes range from 0.08 to 0.66 and all show well-defined periods ranging from 6.50 to 15.5 seconds. Amplitudes of the cyclical components associated with the data range from 0.10 to 0.72 foot, and their relative quantitative significance to the total sea surface roughness patterns are revealed by the terminal amplitudes (r_T). Five of the records show r_T values

TABLE 3

RECORD NUMBER	ORIGINAL DATA					RESIDUAL DATA		
	σ_y^2	$\frac{AD}{\sigma}$	T	A	rT	σ_r^2	$\frac{AD}{\sigma}$	A'
B-713a	0.492	0.78	8.75	0.72	0.56	0.216	0.78	0.74
B-12a	0.018	0.80	9.07	0.10	0.22	0.014	0.83	0.09
53-W	0.109	0.80	15.50	0.35	0.57	0.047	0.79	0.35
53-X	0.185	0.84	12.25	0.51	0.66	0.064	0.77	0.49
53-Y	0.070	0.82	11.00	0.10	0.08	0.065	0.80	0.10
56-E	0.128	0.80	6.50	0.38	0.56	0.057	0.80	0.38
39-L	0.049	0.82	8.29	0.15	0.25	0.035	0.80	0.15
I-5B	0.030	0.84	8.00	0.19	0.63	0.011	0.83	0.19

Results of auto correlation analyses of ocean wave records. σ_y^2 = variance original data, AD/σ = ratio of average deviation to standard deviation, T = period of correlogram, A = amplitude of cyclical component by harmonic analysis, σ_r^2 = variance residual data, r_T = terminal amplitude of original correlogram, $A' = \sqrt{2(\sigma_y^2 - \sigma_r^2)}$.

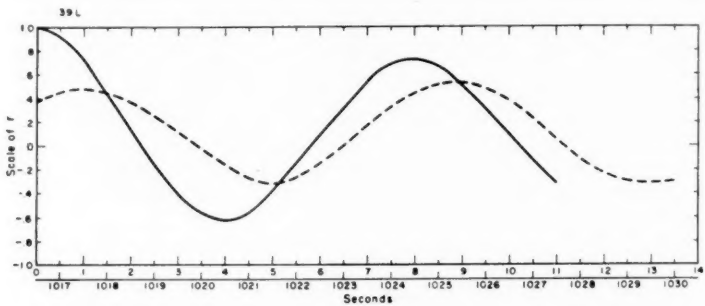


FIGURE 4

Correlogram of Wave Record 39-L, 0 to 11 seconds (solid line) and 1017 to 1030 seconds (dashed line). Curves have been drawn through values of r computed for each second. The curves have not been smoothed.

exceeding 0.5, indicating more than 50% of the roughness pattern resulting from the cyclical component. For the remainder, local influences dominate the pattern, and in one of these, 53-Y, the terminal amplitude of the correlogram is so low (0.08) as to indicate the data to be almost autoregressive. In the complete absence of sea swell the original correlogram damps rapidly to zero.[10]

The results are typical of those obtained from the analyses of more than 30 wave records from the North Atlantic, Mediterranean and the eastern American coast. After extraction of the cyclical components, the residual variances are diminished, and the correlograms of the residuals damp more rapidly in the fashion of an autoregressive series (Fig. 2).

A numerical result of apparent significance is that the ratios of the mean deviations to the standard deviations (for both original and residual data) is approximately equal to $\sqrt{2/\pi}$, a relationship which usually characterizes unimodal curves approaching symmetry. This relationship enables an approximate estimate of the mean sea surface roughness (expressed as twice the average deviation of the original data) to be made directly from the variance of the original data.

The properties of the sea surface roughness pattern, as brought out by auto correlation analysis of a finite amount of data, describe its physical characteristics. They indicate that the instantaneous sea surface pattern is not an interference pattern composed of many wave frequencies which have traveled with independent velocities from their generation areas, such as may be concluded from results of a cycle analysis of finite amounts of data, but rather that it seems to consist of a single cyclical component (sea swell) on which is superimposed random frequencies (local sea) of the same average frequency. The sea swell appears to have been generated by a dominating meteorological situation at some distance from the target and is maintained by energy supplied at a rate equal to that dissipated by the system. It is to be expected that this period changes with time and at a rate dependent on the stability of the off-shore meteorological situation causing it. On the other hand, the local sea is induced by local winds and other factors tending to disturb the sea surface and dies out fairly rapidly.

It may thus be reasoned that the ocean itself acts as a filter which, under specific conditions, passes only certain fundamental frequencies, which we call sea swell. This fundamental frequency at any time is perhaps a function of the characteristics of the ocean basin over which it has traveled, plus wind velocity, direction, fetch and other properties of an established wind-generating system. Its constancy is related to the stability of the meteorological situation. At any particular time local conditions may also produce a continuous band of frequencies with an average frequency the same as that of the sea swell. These latter die out rapidly in a manner similar to the effect of random impulses on a sluggish oscillator as the ocean basin continues to act as a filter.

However, regardless of the exact nature of the physical explanation of the mechanism maintaining the state of the sea, a quantitative separation into sea swell and local sea is fundamental to an understanding of its physical properties. This has an added practical significance for operational purposes as well as for aerial depth determination and for controlled experi-

ments involving sea surface properties. To this end, the method of auto correlation analysis is an improvement over other procedures for the analysis of finite amounts of data in which not even the length of the periods or the recurrence intervals are known or suspected at the outset. The results in themselves provide a more realistic geophysical explanation of the phenomena and avoid the pitfalls from claims of large number of periods and cycles which may arise from procedures less adapted to the available data.

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† The methods and conclusions in this paper have partly resulted from a collaboration with Dr. George P. Wadsworth, of the Massachusetts Institute of Technology, who is preparing a theoretical account of the adaptations of generalized harmonic analysis to the specific problems of ocean waves.

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[5] See also: Kendall, Maurice G., "The Advanced Theory of Statistics," London, 1948; Bartlet, M. S., *et al.*, "Symposium on Auto Correlation in Time Series," *Supp. J. Royal Statistical Soc.*, **VIII**, No. 1, 27–97 (1946).

[6] Seiwell, H. R., "Results of Research on Surface Waves of the Western North Atlantic," papers in *Phys. Ocean. Meteor.*, **X**, No. 4, 1–56 (1948).

[7] The ratio of amplitudes of instantaneous pressure fluctuations at the sea bottom ΔP_h to those at the sea surface ΔP_s is

$$\frac{\Delta P_h}{\Delta P_s} = 1.35 \frac{1}{\text{Cosh } kh}$$

where h is depth to bottom and $k = 2\pi/\text{wave-length}$. The factor 1.35 has been experimentally determined for the Woods Hole–Bermuda area.

[8] These analyses were performed on the ocean wave analyzer at this Institution. The term mechanical periodogram analysis is used to distinguish it from other types of periodogram analyses. See reference 6.

[9] Correlograms for Record 53-X have been published in reference 1.

[10] Examples of this situation have been encountered, although they are not discussed in this paper.

THE CLASS NUMBER OF THE CYCLOTOMIC FIELD

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PRINCETON UNIVERSITY AND INSTITUTE FOR ADVANCED STUDY

Communicated by Marston Morse, July 21, 1949

1. Let g denote any odd prime and $h = h(g)$ the class number of the cyclotomic field $R(\zeta)$, where ζ is the primitive gth root of unity, R the rational numbers. It is known that we can write: $h = h_1h_2$, where h_1 and h_2 (both integers) are the so-called first and second factors of the class number; in fact h_2 is the class number of the real field of degree 2 under $R(\zeta)$, namely the field $R(\zeta + \zeta^{-1})$.

Kummer conjectured (*J. Math.*, **16**, 473 (1851)) that

$$h_1 \sim \frac{g^{(g+3)/4}}{2^{(g-3)/2}\,\pi^{(g-1)/2}} = G. \tag{1}$$

He also calculated h_1 for $g \leqq 97$ and found $h_1 = 1$ for $g \leqq 19$, $h_1 = 411322823001$ for $g = 97$. No proof of (1) has yet been published.

In this paper we prove the following result:

$$\log\left(\frac{h_1}{G}\right)/\log g \to 0 \qquad (g \to \infty). \tag{2}$$

An interesting consequence of (2) is that there exists a g_0 such that $h_1(g)$ is (strictly) monotonic increasing for $g > g_0$; in fact if $g_2 > g_1 > g_0$, we have

$$h_1(g_2) > h_1(g_1).$$

Let p stand for a typical prime. Define

$$c_p = +1, -1, 0 \qquad [p \equiv 1(g),\ p \equiv -1(g),\ p \not\equiv \pm 1(g)]$$

It is well known that the series

$$\Sigma c_p \cdot p^{-1}$$

is convergent. We can show that *if Kummer's conjecture is true, we must have:*

$$\Sigma c_p \cdot p^{-1} = O(g^{-1}) \qquad [g \to \infty]. \tag{3}$$

It is hard to say whether (3) is even probably true. We cannot prove (3) even on the assumption of the "extended Riemann hypothesis" (or e. R. h.).

The method used to prove the above results can also be applied to prove the following results:

Let θ_1 and θ_2 denote any fixed constants such that

$$1/2 < \theta_1 < s < \theta_2 < 1.$$

If the e. R. h. is true, there exists for every given $\epsilon > 0$, a non-principal character $X(n) \pmod g$ such that

$$|L(s)| = |\sum_1^\infty X(n) \cdot n^{-s}| < 1 + \epsilon \tag{4}$$

for all primes $g > g_0(\epsilon, \theta_1, \theta_2)$.

This result with the larger constant $\sqrt{\zeta(2s)} + \epsilon$ on the right hand side of the inequality (4) (here, $\zeta(s)$ denotes Riemann's Zeta function; hence $\zeta(2s) > 1$ for $s > 1/2$) is implicit in some recent work of Atle Selberg, but it (with the larger constant) is proved by him without any hypothesis (*Skrifter Norske Videnskaps-Akad. Oslo*, Nr. 3 (1946)).

2. We give only a sketch of the methods used to prove the above results. Complete proofs will be published elsewhere.

Notation: We write

$$a \equiv b(c) \text{ for } a \equiv b \pmod c$$

Let X_t denote the $1/2(g-1)$ characters (mod g), such that $X_t(-1) = -1$. $\pi(x; k, l)$ denotes the number of primes $\equiv l(k)$ and not exceeding x; $\phi(k)$ is the number of positive integers not exceeding k and prime to k.

$X_t(n)$ is a non-principal character (mod g), hence

$$L_t(s) = \sum_1^\infty X_t(n) \cdot n^{-s} \qquad [R(s) > 0]$$

It now easily follows that for $s > 1$ we have

$$\prod_t L_t(s) = \exp\left\{g' \sum_p \sum_{m \geq 1} \frac{c_{p,m}}{m p^{ms}}\right\} \tag{5}$$

where $g' = 1/2(g-1)$ and

$$c_{p,m} = +1, -1, 0 [p^m \equiv 1(g), p^m \equiv -1(g), p^m \not\equiv \pm 1(g)].$$

From (5) and

$$\frac{h_1}{G} = \prod_t L_t(1) \tag{6}$$

we deduce that

$$\frac{h_1}{G} = \exp\left\{g' \sum \frac{c_p}{p} + g' \sum p \sum_{m>2} \frac{c_{p,m}}{m p^m}\right\} \tag{7}$$

Write

$$C = \sum c_p \cdot p^{-1} = C_1 + C_2 + C_3 \tag{8}$$

where for any given $\epsilon > 0$:

C_1 is the contribution to C of the terms with $2g - 1 \leqq p < g^{1+\epsilon}$

C_2 is the contribution to C of the terms with

$$g^{1+\epsilon} < p < \exp(g^{2\theta});$$

C_3, the rest.

It is trivial that (B is an absolute positive constant)

$$|C_1| \leqq B \frac{\epsilon}{g} \log g. \tag{9}$$

Now write

$$C(x) = \sum_{p \leqq x} C_p$$

Then the sum C_2 is essentially

$$\sum \frac{C(n)}{n(n+1)} \tag{10}$$

where the summation is for all integers n lying between

$$g^{1+\epsilon} \text{ and } \exp(g^{2\gamma}).$$

To deal with the sum (10) we use the following result:

$$\pi(x;\ k,\ l) = O\left\{\frac{x}{\phi(k) \log x}\right\} \qquad [x \geqq k^{1+\epsilon}]. \tag{11}$$

This result follows from the work of Viggo Brun, but was first implicitly stated by Titchmarsh (*Rendiconti Circolo Mat. Palermo*, **54**, 414–429 (1930)). The constant implied in the O symbol is independent of x and k but may depend on ϵ.

From (11) one easily deduces that

$$|C_2| \leqq \frac{A\theta K(\epsilon) \log g}{g} \tag{12}$$

where A is an absolute positive constant.

Finally to estimate C_3 we use the following theorem (*Walfisz, Math. Ztschr.*, **40**, 598–599 (1936)): If $(k, l) = 1$, then

$$\pi(x;\ k,\ l) - \frac{1}{\phi(k)} \int_2^x \frac{du}{\log u} = O\left\{\frac{x}{\log^m x}\right\} + O\left\{\frac{x^{1 - ck^{-\theta}}}{\phi(k) \log x}\right\}$$

where the constant implied in the O symbol is independent of x and k, but may depend on m (arbitrarily large) and θ (arbitrarily small). By the application of this result we can easily obtain:

$$C_3 = O[g^{(1-m)2\theta} + g^{-1-2\theta}] \tag{13}$$

We first fix ϵ (small), then θ sufficiently small, and finally m sufficiently large, to obtain from (8), (9), (12) and (13):

$$\left(\frac{g}{\log g}\right) \Sigma c_p \cdot p^{-1} \to 0 \ (g \to \infty) \tag{14}$$

We next indicate the proof of

$$C_4 = \sum_p \sum_{m \geq 2} \frac{c_{p,m}}{mp^m} = O(g^{-1}) \tag{15}$$

(14) shows that (2) follows from (15). In fact,

$$|C_4| \leqq \sum_p \sum_{m \geq 2} \frac{1}{mp^m} \qquad [p^m \equiv \pm 1(g)]$$

The part of this sum with $p > g$ is shown to be $O(g^{-1})$ by a very short argument.

Since $X^m \equiv 1(g)$ has at most m incongruent solutions, we derive that

$$C_4 = O\left\{ \sum_{m=2}^{\infty} \frac{1}{m} \left[\sum_{a=A(m)}^{B(m)} \left(\frac{1}{ag+1} + \frac{1}{ag-1} \right) \right] \right\}$$

where

$$A(m) = {}^1/_2(m^2 - m) + 1, \qquad B(m) = {}^1/_2(m^2 + m)$$

And this indicates the proof of (15).

To prove (4) we start from the result

$$C(X) = O(\sqrt{x} \log x)$$

which is true on the assumption of the e. R. h. The rest of the arguments are similar to those used above.

Also, on the assumption of the e. R. h. we can improve (2) to

$$\log \left(\frac{h_1}{G}\right) = O\{(\log g)^{1/2} \log \log g\}$$

SOME PROPERTIES OF A SFIELD

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All the results of this paper are initiated from the almost trivial identity that if $ab \neq ba$, then

$$a = (b^{-1} - (a-1)^{-1}b^{-1}(a-1))(a^{-1}b^{-1}a - (a-1)^{-1}b^{-1}(a-1))^{-1} \tag{1}$$

Among these results, there are two interesting ones which are the perfect form of two theorems due to H. Cartan[1] and J. Dieudonné,[2] respectively.

THEOREM 1. *Every sfield is generated by a non-central element and its conjugates.*

Let K be the sfield and L be the conjugate set. The identity (1) asserts that if there is an element b of L such that $ab \neq ba$, then a belongs to the sfield K, generated by L. That is, $K - K_1$ and K_1 are commutative elementwise.

Suppose our theorem is false; we have an element a of $K - K_1$ and two elements b and b' of K_1 such that $bb' \neq b'b$. Since a and ab belong to $K - K_1$, we have $(ab)b' = b'(ab) = ab'b$, which contradicts $bb' = b'b$. The theorem follows.

An immediate consequence of Theorem 1 is the following interesting result:

THEOREM 2. *Every proper normal subsfield of a sfield is contained in the center.*

H. Cartan proved the theorem under the assumption that the rank of the sfield over its center is finite. His proof is far more complicated than the present one.

More precise results can be obtained by specializing the subgroup.

THEOREM 3. *Every sfield, which is not a field, has no proper subsfield containing all its commutators.*

It is enough to prove the existence of a commutator which does not belong to the center. Suppose $ab \neq ba$. From a modification of the identity (1):

$$a = (1 - (a-1)^{-1}b^{-1}(a-1)b)(a^{-1}b^{-1}ab - (a-1)^{-1}b^{-1}(a-1)b)^{-1}, \tag{2}$$

we deduce that at least one of $a^{-1}b^{-1}ab$ and $(a-1)^{-1}b^{-1}(a-1)b$ does belong to the center.

As corollaries of Theorem 3, we have

THEOREM 4. *If the center of a sfield contains all the commutators of the*

sfield, then the sfield is commutative. An element of a sfield commutative with all the commutators of the sfield belongs to the center.

Moreover, as a simple consequence of

$$a^{-1}c^{-1}ac = a^{-2}(ac^{-1})^2c^2 \tag{3}$$

and Theorem 3, we have

THEOREM 5. *Every sfield, which is not a field, has no proper subsfield containing all its square elements.*

THEOREM 6. *If the center of a sfield contains all the square elements of the sfield, then the sfield is commutative. An element of a sfield commutative with all the square elements of the sfield belongs to the center.*

By means of an identity in the theory of finite differences:

$$\Delta^{k-1}x^k = k!(x + {}^1/_2(k - 1)),$$

where $\Delta f(x) = f(x + 1) - f(x)$ and $\Delta^i f(x) = \Delta(\Delta^{i-1}f(x))$, we have

THEOREM 7. *Every sfield of characteristic $\geq k$, which is not commutative, has no proper subsfield containing all its k-th power elements.*

Let us denote $a^{-1}b^{-1}ab$ by (a, b). An r-commutator ($r \geq 2$) is defined inductively by the relation

$$(a_1, \ldots, a_r) = (a_1, (a_2, \ldots, a_r)). \tag{4}$$

Let C_r be the set of all r-commutators of the sfield.

THEOREM 8. *Let $r \geq 2$. Every sfield, which is not commutative, has no proper subsfield containing all its r-commutators. An element of a sfield which permutes with all the r-commutators of the sfield belongs to the center.*

Let A_r and B_r denote the two propositions of our theorem. B_r follows from A_r immediately. Theorems 3 and 5 assert that both A_2 and B_2 are true. We shall prove the theorem by induction, we assume that A_{r-1} and B_{r-1} are both true. Let a be an element of the sfield, which does not belong to the center. By B_{r-1}, we have an element belonging to C_{r-1} such that $ab \neq ba$. The identity (2) asserts that a belongs to the sfield generated by C_r. Therefore A_r is true. The theorem is proved.

The theorem asserts that the lower central series of the multiplicative group of a sfield can never be ended at the identity. It is an almost trivial fact that the upper central series can never go up. It is a comparatively difficult question about the derived series. Elsewhere I shall prove

THEOREM 9. *Let $r \geq 1$. Every sfield is generated by all the elements of the r-th derived group.*

An application of Theorem 1 is to establish the simplicity of the projective special linear group $PSL_n(K)$[3] of dimension n over the sfield K.

THEOREM 10. *The projective special linear group $PSL_n(K)$ is simple except when $n = 2$ and K has 2 or 3 elements.*

In case K is a field of characteristic $\neq 2$ or K is a complete field, the

theorem was proved by Burnside-Jordan-Dickson (see Dickson,[4] p. 85 and Van der Waerden,[5] p. 7). For the incomplete field, it was proved by Iwasawa.[6] For a sfield, Dieudonné[2] proved the theorem for $n > 2$, and for $n = 2$[7] except in the case when the center of K has 2, 3 or 5 elements. Since both exceptional cases in Theorem 10 are not simple, it is the final answer to the problem about the simplicity of $PSL_n(K)$.

Owing to the previous known results, we assume, from now on, that $n = 2$ and the sfield is not commutative. It is the same thing to prove that

THEOREM 11. *Every normal subgroup of $SL_2(K)$ is contained in the center, except when K has 2 or 3 elements.*

Let A be an element of $SL_2(K)$ which does not belong to the center. We are going to prove that A and its conjugates generate $SL_2(K)$. We shall prove that there is a matrix Q belonging to $SL_2(K)$ such that QAQ^{-1} has a non-zero element at (1, 2) position. In fact, let $A = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. If $\beta \neq 0$, $Q = I$ satisfies our requirement, and if $\gamma \neq 0$, then $Q = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ does. If $\beta = \gamma = 0$, we have an element x such that $\alpha x \neq x\delta$, since A does not belong to the center. Then

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} * & -\alpha x + x\delta \\ * & * \end{pmatrix}.$$

We may assume that the subgroup generated by A and its conjugates contains an element with $\beta \neq 0$. Since

$$\begin{pmatrix} 1 & 0 \\ \beta^{-1}\alpha & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \beta^{-1}\alpha & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & * \\ * & * \end{pmatrix},$$

we have assumed that the group contains an element of the form $A = \begin{pmatrix} 0 & \beta \\ \gamma & \delta \end{pmatrix}$. We take

$$B = \begin{pmatrix} -\gamma^{-1}\delta\beta^{-1} & \gamma^{-1} \\ -\gamma\kappa & 0 \end{pmatrix},$$

where κ belongs to the commutator subgroup of K, so that B belongs to $SL_2(K)$. We have

$$A(BA^{-1}B^{-1}) = AB(BA)^{-1} = \begin{pmatrix} -\beta\gamma\kappa & 0 \\ * & -(\gamma\kappa\beta)^{-1} \end{pmatrix}.$$

Now we shall prove that we can choose κ such that $-\beta\gamma\kappa$ does not belong to the center. For otherwise $(-\beta\gamma\kappa_1)^{-1}(-\beta\gamma\kappa_2) = \kappa_1^{-1}\kappa_2$ belongs to the center for any two commutators κ_1 and κ_2, that is, the commutators of κ belong to the center of K. By Theorem 4, K is a field which contradicts our supposition. Therefore the subgroup contains an element of the form $C = ABA^{-1}B^{-1} = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$, where a does not belong to the center. We

have an element λ of K such that $d\lambda \neq \lambda a$, then the subgroup contains

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ d\lambda a^{-1} - \lambda & 1 \end{pmatrix}.$$

We may suppose that our subgroup contains $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$, $c \neq 0$.

We are going to prove that the subgroup contains $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. In fact, if $c = \pm 1$, our assertion is trivial. Let us suppose $c \neq 0, \neq \pm 1$. The field K_1, obtained from the center of K by adjunction of c, contains more than 3 elements. $SL_2(K_1)$ is generated by any one element, not in the center, and its conjugates; in particular, $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and its conjugates generate $SL_2(K_1)$. Thus $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is in the subgroup under consideration. Consequently the subgroup contains

$$\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ \lambda^2 & 1 \end{pmatrix}$$

for all λ.

If the characteristic of the sfield is not equal to 2, then the subgroup contains all the elements of the form

$$\begin{pmatrix} 1 & 0 \\ (\lambda + 1)^2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda^2 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 2\lambda & 1 \end{pmatrix}.$$

Therefore we have the theorem.

If the characteristic of the sfield equals 2, since

$$\begin{pmatrix} 1 & (\lambda + 1)^{-2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \lambda^2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (1 + \lambda^{-1})^{-2} & 1 \end{pmatrix} = \begin{pmatrix} (\lambda + 1)^{-2} & 0 \\ 0 & (\lambda + 1)^2 \end{pmatrix},$$

the subgroup contains elements of the form

$$\begin{pmatrix} a^2 & 0 \\ 0 & a^{-2} \end{pmatrix}.$$

for all a. Let a be any element of the sfield. From

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^2 0 \\ 0 & a^{-2} \end{pmatrix}\begin{pmatrix} 1 & \lambda^2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} a^{-2} & 0 \\ 0 & a^2 \end{pmatrix} = \begin{pmatrix} 1 & a^2\lambda^2 a^2 + a^5 + a \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a^2 0 \\ 0 & a^{-2} \end{pmatrix}\begin{pmatrix} 1 & \lambda^2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^2 0 \\ 0 & a^{-2} \end{pmatrix}^{-1}\begin{pmatrix} (a + 1)^2 & 0 \\ 0 & (a + 1)^{-2} \end{pmatrix} \times$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (a + 1)^2 & 0 \\ 0 & (a + 1)^{-2} \end{pmatrix}^{-1},$$

we deduce that $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ belongs to the subgroup. Since $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and its conjugates generate $SL_2(K)$ as a runs over all elements of K, the theorem follows.

The previous method can be used to establish the easier

THEOREM 12. *Every normal subgroup of the general linear group $GL_n(K)$, which is not contained in the center, contains $SL_n(K)$, except the cases when $n = 2$ and K has 2 or 3 elements.*

[1] Cartan, H., *Ann. école normale superieure*, **64**, 59–77 (1947), Theorem 4.

[2] Dieudonnè, J., *Bull. Soc. math. France*, **71**, 27–45 (1943).

[3] For the definition and properties of $PSL_n(K)$, see Dieudonnè.[6]

[4] Dickson, L. E., *Linear Groups*, Leipzig, 1901.

[5] Van der Waerden, B. L., *Gruppen von linearen Transformationen*, Berlin, 1935, p. 7.

[6] Iwasawa, M., *Proc. Imp. Acad. Japan*, **17**, 57 (1941).

[7] The author had some difficulty in understanding Dieudonné's proof. In fact, all the parabolic elements of $PSL_2(K)$ do not form a single conjugate set in $PSL_3(K)$.

IMPRIMITIVITY FOR REPRESENTATIONS OF LOCALLY COMPACT GROUPS I

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1. Introduction.—In the classical theory of representations of finite groups by linear transformations a representation $s \to U_s$ of a finite group is said to be imprimitive if the vector space H in which the U_s act is a direct sum of independent subspaces $M_1, M_2, \ldots, M_n$ in such a manner that each U_s transforms each M_i into some M_j. In the present note we shall discuss a generalization of this notion which is more suitable for use in connection with infinite dimensional representation because it allows the direct sum decomposition to be continuous as well as discrete. Our principal theorem (well known for finite groups) deals with weakly (and hence strongly) continuous unitary representations of separable locally compact groups. It asserts that the pair consisting of such a representation and a "transitive system of imprimitivity" for it defines an essentially unique subgroup G_0 and an essentially unique representation L of G_0 from which the original pair may be reconstructed in a quite explicit manner. This result has a number of applications. A recent theorem[1] of the author which implies the Stone-von Neumann theorem on the uniqueness of operators satisfying the Heisenberg commutation relations is included as a special case. In addition it may be used to give a complete determi-

nation of the irreducible unitary representations of the members of a class of locally compact groups which are neither compact nor Abelian.

2. Definition of Imprimitivity.—Let $s \to U_s$; $M_1, M_2, \ldots, M_n$ be an imprimitive representation in the classical sense. Suppose that the U_s are unitary and that the M_i are mutually orthogonal. Let M denote the set of integers $1, 2, \ldots, n$. For each s in the group G and each $j \in M$ let $(j)s$ be the index of the subspace into which U_{s-1} carries M_j. Let P_j denote the projection of H on M_j. Then it is easy to see that $U_s P_j U_s^{-1} = P_{(j)s^{-1}}$. More generally if P_E is defined by the equation $P_E = \sum_{(j \in E)} P_j$ for each $E \subseteq M$ then $U_s P_E U_s^{-1} = P_{(E)s^{-1}}$. The motivation for the following definition should now be clear. Let M be a separable locally compact space and let G be a separable locally compact group. Let $x, s \to (x)s$ denote a mapping of $M \times G$ onto M which is continuous and is such that (a) for fixed s, $x \to (x)s$ is a homeomorphism and (b) the resulting map of G into the group of homeomorphisms of M is a homomorphism. Let $P(E \to P_E)$ be a σ homomorphism of the σ Boolean algebra of all Borel subsets of M into a σ Boolean algebra of projections in a separable Hilbert space H such that P_M is the identity I. Let $U(s \to U_s)$ be a representation of G in H; that is a weakly (and hence strongly) continuous homomorphism of G into the group of unitary operators in H. If $U_s P_E U_s^{-1} = P_{(E)s^{-1}}$ for all E and s and if P_E takes on values other than O and I we shall say that U is imprimitive and that P is a system of imprimitivity for U. We shall call M the base of P. It is to be observed that P defines in M a family of null sets and that there exists in M a family of mutually equivalent measures whose sets of measure zero are precisely these null sets. The null sets are those sets E for which $P_E = 0$ and the measures are those of the form $\mu(E) = (P_E f, f)$ where f is an element[2] in H such that $P_E f = 0$ implies $P_E = 0$.

3. Ergodicity and Transitivity.—When for each x and y in M there exists s in G for which $(x)s = y$ it is natural to say that P is a transitive system of imprimitivity for U. When M is finite every system of imprimitivity decomposes in a natural manner into transitive ones corresponding to the orbits of M under G. In general, however, the decomposition of M into orbits is not reflected in a corresponding decomposition of H. It is rather the decomposition of M into ergodic or metrically transitive parts which is relevant. Accordingly we define a system of imprimitivity P to be ergodic if G acts ergodically on the base M of P; that is, whenever $(E)s$ differs from E by a null set for all $s \in G$ then E is itself a null set or the complement of one. In view of the current literature on the decomposition of measures the study of general systems of imprimitivity may be expected to be reducible to the study of ergodic systems.

Ergodic systems which are not also transitive are rather difficult to handle and such results as we have at present are far from definitive. This

note will deal exclusively with transitive systems. Fortunately in some applications it can be shown that only transitive systems can arise. Specifically let us say that the orbits of M under G are regular if there exists a countable family $E_1, E_2, \ldots$ of Borel subsets of M, each a union of orbits such that each orbit of M is the intersection of the members of a subfamily $E_{n_1}, E_{n_2}, \ldots$. Then the following theorem is easily proved.

THEOREM 1. *If the orbits of M under G are regular then for each ergodic system of imprimitivity based on M there is an orbit C such that $P_{M-C} = 0$.*

4. *Formulation of the Principal Theorem.*—Let P be a transitive system of imprimitivity for the representation U of the separable locally compact group G. Let x_0 be a point of the base M of P. Let G_0 be the set of all $s \in G$ for which $(x_0)s = x_0$. Then G_0 is a closed subgroup of G and the mapping $s \to (x_0)s$ of G on M defines a one-to-one Borel set preserving map of the homogeneous space G/G_0 of right G_0 cosets onto M. Thus P is equivalent in an obvious sense to another system of imprimitivity for U whose base is the homogeneous space G/G_0. In general we shall define a pair to be a unitary representation for the group G together with a particular system of imprimitivity for this representation. If U, P and U', P' are two pairs with the same base M we shall say that they are unitary equivalent if there exists a unitary transformation V from the space of U and P to the space of U' and P' such that $V^{-1}U_s'V = U_s$ and $V^{-1}P_E'V = P_E$ for all s and E. It follows from the above remarks that the problem of determining to within unitary equivalence all pairs based on a given M may always be reduced to the corresponding problem in which M is a homogeneous space. We shall accordingly confine ourselves to this case. The arbitrariness in the choice of x_0 has the effect only of providing us with several essentially equivalent complete systems of invariants for the pairs based on a given M.

Preparatory to stating our theorem we describe a method (which will prove to be general) of constructing pairs based on a given G/G_0. Let μ be a finite Borel measure on G/G_0 which is "quasi invariant" in the sense that the action of G on G/G_0 preserves null sets.[3] Let $L(\xi \to L_\xi)$ be a representation of G_0 by unitary operators in a Hilbert space H_0. Then let H_L be the set of all functions f from G to H_0 such that: (*a*) f is a Borel function in the sense that $(f(s), v)$ is a Borel function of s for all $v \in H_0$; (*b*) for all $s \in G$ and all $\epsilon \leftarrow G_0$, $f(\xi s) = L_\xi f(s)$; and (*c*) $(f(s), f(s))$ (which by (*b*) is constant on the right G_0 cosets) defines a summable function on G/G_0. By a more or less obvious adaptation of the proof of the Riesz Fischer theorem[4] it may be shown that H_L is a Hilbert space with respect to the inner product $(f, g)_L = \int_{G/G_0}(f(s), g(s))d\mu$ and the obvious linear operations. Naturally functions which are equal almost everywhere are to be identified. Now let ρ be the function on $G \times G/G_0$ which for each fixed s is the Radon Nikodym derivative of the translate of μ by s with

respect to μ itself. Then regarding ρ, as we may, as a function on $G \times G$ let $U_s f$ for all $s \in G$ and $f \in H_L$ be defined by the equation $(U_s f)(t) = f(ts)/\sqrt{\rho(s^-, ts)}$. It is readily verified that U_s is a unitary transformation of H_L onto itself and that the mapping $s \to U_s$ is a representation of G. For each Borel subset E of G/G_0 let ϕ be its characteristic function regarded as a function on G. For $f \in H_L$ let $(P_E f)(t) = \phi(t)f(t)$. It is easy to see that the mapping $f \to P_E f$ is a projection and that U and P together constitute a pair in the sense of the above definition. We shall call it the pair generated by L and μ. We can now formulate our main theorem.

THEOREM 2. *Let G be a separable locally compact group and let G_0 be a closed subgroup of G. Let U', P' be any pair based on G/G_0. Let μ be any quasi invariant measure in G/G_0. Then there exists a representation L of G_0 such that U', P' is unitary equivalent to the pair generated by L and μ. If L and L' are representations of G_0 and μ and μ' are quasi invariant measures in G/G_0 then the pair generated by L' and μ' is unitary equivalent to the pair generated by L and μ if and only if L and L' are unitary equivalent representations of G_0.*

5. *Outline of Proof.*—We shall give the proof in outline only leaving relatively routine details to the reader. Moreover we shall assume familiarity on the part of the reader with the paper cited in reference 1 and will omit arguments similar to those given there. We shall refer to this paper as SVN. The proof falls naturally into two parts. First we show that every pair defines a representation of G_0 unique to within unitary equivalence and that two pairs defining equivalent representations of G_0 are unitary equivalent. Then we complete the proof by showing that the representation of G_0 defined by the pair generated by an arbitrary L and μ is unitary equivalent to L itself.

Given a pair U', P' based on G/G_0 we note first that the set of all P_E' is a uniformly n dimensional Boolean algebra of projections ($n = 1, 2, \ldots, \infty$) in the sense of Nakano (see SVN 5). This follows from the fairly easily proved fact that G acting on G/G_0 is ergodic. Let N denote an n dimensional identity representation of G_0, let μ be a quasi invariant measure in G/G_0 and let W,P be the pair generated by N and μ. Just as in No. 6 of SVN it is possible to show that the pair U',P' is unitary equivalent to the pair U,P where P comes from the pair W,P above and U is a suitable representation of G. We define Q_s as $U_s W_s^{-1}$ and observe that $Q_s P_E = P_E Q_s$ for all E and s. It follows as in SVN that there exists a weakly Borel function $Q^\sim$ from $G \times G$ to the group of unitary operators in the space H_1 in which the N_s act such that for each s in G we have $(Q_s f)(t) = Q^\sim(s, t)f(t)$. The identity $Q^\sim(s_1 s_2, t) = Q^\sim(s_1, t)Q^\sim(s_2, ts_1)$ holding for almost all triples is established as in SVN and from it the existence of a weakly Borel function B such that $Q^\sim(s, t) = B^{-1}(t)B(ts)$ almost everywhere. The fact that the functions in H_N are constant on the

right G_0 cosets implies that $Q^\sim(s, \xi t) = Q^\sim(s, t)$ for all $\xi \epsilon G_0$ almost everywhere in s and t. This implies in turn that $B^{-1}(\xi t)B(\xi ts) = B^{-1}(t)B(ts)$ in the same sense or equivalently that $B(\xi ts)B^{-1}(ts) = B(\xi t)B^{-1}(t)$. In short for each $\xi \epsilon G_0$, $B(\xi t)B^{-1}(t)$ is almost everywhere equal to a certain constant operator L_ξ. A simple argument shows that $(L_\xi v_0, v_1)$ is of the form $\int_G \psi(t)(B(\xi t)v_2, v_1)$ for a dense set of v_0's. Here v_0, v_1 and v_2 are elements in H_1 and ψ is a continuous complex valued function vanishing outside of a compact subset of G. It follows readily that $(L_\xi v_0, v_1)$ is continuous in ξ and, since $L_{\xi_1\xi_2} = L_{\xi_1}L_{\xi_2}$, that $L(\xi \to L_\xi)$ is a representation of G_0. Of course L may depend upon the choice of μ, the choice of the unitary map of the given Hilbert space on H_N and the choice of B. However, the fact that any two μ's have the same null sets guarantees the lack of dependence of L on μ. As to the other possible dependencies note that a unitary map X of H_N on itself which commutes with all P_E is defined by an equation of the form $Xf(t) = X(t)f(t)$ where $X(t)$ is a unitary operator on H_1 for each t and $X(t)$ is a weakly Borel function of t. Moreover $X(\xi t) = X(t)$ for $\xi \epsilon G_0$. It is readily calculated that the effect on Q of a transformation by X is to replace it by R where $R(s, t) = X^{-1}(t)Q(s, t)X(ts)$. Now if $C^{-1}(t)C(ts) = X^{-1}(t)B^{-1}(t)B(ts)X(ts)$ it follows that $B(t)X(t)C^{-1}(t)$ is (modulo null sets) independent of t. Thus for some constant operator K we have $C(t) = KB(t)X(t)$ so that $C(\xi t)C^{-1}(t) = KB(\xi t)X(\xi t)X^{-1}(t)B^{-1}(t)K^{-1} = KL_\xi K^{-1}$. In short our original pair and in fact the unitary equivalence class to which it belongs determine L to within unitary equivalence. Conversely a simple reversal of the argument shows that pairs leading to unitary equivalent L's must be themselves unitary equivalent.

Now let L' be an arbitrary representation of G_0 and let U',P' be the pair generated by L' and a quasi invariant measure μ in G/G_0. By the argument of the preceding paragraph there is a unitary map V^{-1} of $H_{L'}$ on some H_N such that $V^{-1}P_E'V = P_E$ where W,P is as before the pair generated by N and μ and N is an identity representation of G_0 on a Hilbert space H_1. It is not difficult to show that there exists a weakly Borel function $V^\sim$ defined on G whose values are operators from H_1 to the space H_2 in which L' operates such that $(Vf)(t) = V^\sim(t)f(t)$. It follows from the fact that V is unitary that $V^\sim(t)$ is unitary from H_1 *into* H_2 for almost all t and it follows from the fact that $Vf \epsilon H_{L'}$ that for each $\xi \epsilon G_0$, $V^\sim(\xi t) = L_\xi' V^\sim(t)$ for almost all t. Now the Q_s of the preceding paragraph here take the form $V^{-1}U_s'VW_s^{-1}$ so that $U_s'VW_s^{-1} = VQ_s$. Hence $V^\sim(ts) = V^\sim(t)Q^\sim(s, t)$ or $V^\sim(ts) = V^\sim(t)B^{-1}(t)B(ts)$ or $V^\sim(ts)B^{-1}(ts) = V^\sim(t)B^{-1}(t)$. Thus there exists a norm preserving operator K independent of t such that $V^\sim(t) = KB(t)$ for almost all t. If K were known to map H_1 onto the whole of H_2 we could write $B(t) = K^{-1}V^\sim(t)$ and conclude at once that $B(\xi t)B^{-1}(t) = K^{-1}L_\xi' V^\sim(t)V^{\sim -1}(t)K = K^{-1}L_\xi K$ and hence that the L for U',P' is unitary equivalent to L'. In order to show that K is indeed

an onto mapping we must make use of certain facts about the space $H_{L'}$ which so far as we know at this point might be zero dimensional. For each continuous function w from G to H_2 which vanishes outside of a compact subset of G let $\tilde{w}$ be defined by the equation $(\tilde{w}(t), v) = \int_{G_0} (L_\xi'^{-1}w(\xi t), v)d\xi$ for all v in H_1 and all t in G. This function may be shown to be a continuous member of $H_{L'}$ which vanishes outside of a set whose image in G/G_0 is compact. Arguments of a fairly routine nature show that for each $t \in G$ the vectors $\tilde{w}(t)$ span H_2. Now suppose that K does not map H_1 onto H_2. Choose v_0 orthogonal to the range of K. Consider an arbitrary member of $H_{L'}$ of the form $\tilde{w}$. We have $\tilde{w}(t) = V^\sim(t)f(t)$ for some f and almost all t. But $V^\sim(t)f(t)$ is in the range of K for almost all t. Thus, since $\tilde{w}$ is continuous we can conclude that $(\tilde{w}(t), v_0) = 0$ for all t. Hence for all t, $(\tilde{w}(t), v_0) = 0$ for all $\tilde{w}$ and this contradicts the fact that the $\tilde{w}(t)$ span for each t.

6. *Reducibility.*—The natural question concerning the connection between the reducibility of a pair U,P and the reducibility of the defining representation of G_0 is easily answered. If T commutes with all L_ξ then a transformation $T^\sim$ taking H_N into H_N is defined by the equation $(T^\sim f)(t) = B^{-1}(t)TB(t)f(t)$ where B is the function used in defining L. Then, as is easily seen $T \to T^\sim$ is a $*$-isomorphism of the ring of all bounded linear operators which commute with all the L_ξ onto the ring of all bounded linear operators which commute with all the U_s and all of the P_E. In particular the U_s and the P_E are simultaneously reducible if and only if L is a reducible representation of G_0.

7. *Application to the Determination of Group Representations.*—Let G be a separable locally compact group and let G_1 be a closed normal Abelian subgroup of G. Let $\hat{G}_1$ denote the character group of G_1. Every member s of G defines an automorphism $x \to sxs^{-1}$ of G_1 and this in turn induces an automorphism $y \to (y)s$ of $\hat{G}_1$. Now let U be any irreducible representation of G. Restricted to G_1 it admits a spectral resolution defined by a σ homomorphism P of the Borel subsets of $\hat{G}_1$ into a Boolean algebra of projections in the Hilbert space H in which U acts. An obvious calculation shows that $U_sP_EU_s^{-1} = P_{(E)s^{-1}}$. Thus P is a system of imprimitivity for U. Since U is irreducible P must be ergodic. If we assume that G_1 is "regularly imbedded" in G in the sense that the orbits in G_1 under G are regular then Theorem 1 tells us that G_1 may be replaced by a single orbit. Let y be a point in this orbit and let G_y be the closed subgroup of all s for which $(y)s = y$. Theorem 2 tells us that U is unitary equivalent to the first member of the pair generated by an irreducible representation of G_y.

If G is a "semi-direct product" of G_1 and G/G_1; that is, if there exists a closed subgroup G_2 such that $G_1 \cap G_2 = e$ and $G_1G_2 = G$ much more precise information is available. The reader will have no trouble in verifying the truth of

THEOREM 3. *Let G_1 be imbedded regularly in G and let G be a semidirect product of G_1 and G_2. From each orbit C of G_1 under G_2 choose a member y_C. Let G_C denote the set of all $s \in G_2$ with $(y_C)s = y_C$. Then the general irreducible representation of G may be obtained as follows. Select an orbit C and an irreducible representation L of G_C. Let M be the irreducible representation of $G_1 \cdot G_C$ which coincides with L on G_C and is y_C times the identity on G_1. Then the first member of the pair generated by M and a quasi invariant measure in $G/(G_1 \cdot G_C)$ is the required irreducible representation of G. Every irreducible representation of G may be so obtained and two such are unitary equivalent if and only if they come from the same orbit and unitary equivalent L's.*

When the irreducible representations of G_2 and its subgroups are known Theorem 3 furnishes a complete description of the irreducible representations of G. This is so, in particular, when G_2 is Abelian. Moreover when G_2 is Abelian (and G_1 is imbedded regularly in G) it tells us that every irreducible representation of G is of "multiplier" form. More generally any imprimitive representation of G generated by a one-dimensional representation of a subgroup is unitary equivalent to a representation in which the underlying Hilbert space is the space of square summable functions on a homogeneous space and the action of the operator associated with s is to translate by s and multiply by a certain function (the multiplier) of s and a variable point in the homogeneous space.

When G_1 is not imbedded regularly in G Theorem 3 fails only in that it does not describe all of the irreducible representations. The ones that it does describe still exist and are irreducible. We have examples, however, showing that in general there are many others. Their existence leads to various kinds of pathological behavior which we expect to discuss at another time. Since these "extra" representations are all infinite dimensional, Theorem 3 provides an analysis of all finite dimensional representations for arbitrary semidirect products.

A number of well-known groups are regular semidirect products and Theorem 3 includes as special cases results in the literature analyzing their representations. Examples include the unique non-commutative two-parameter Lie group[5] (a semidirect product of two lines) and the group of Euclidean motions in two space[6] (a semidirect product of the two-dimensional translation group and the circle group). Wigner's[6] reduction of the representation problem for the inhomogeneous Lorentz group to that for the homogeneous Lorentz group is also a consequence of Theorem 3 since the former group is a semidirect product of a translation group and the latter group.

8. Concluding Remarks.—In the mapping from a representation L of G_0 to the pair it generates one can ignore P and obtain a mapping from representations of G_0 to representations of G. It is not difficult to see that

this mapping carries the regular representation of G_0 into the regular representation of G. Thus (in view of No. 6) any analysis of the regular representation of G_0 as a direct sum or integral will define a corresponding analysis of the regular representation of G although the "parts" will not necessarily be irreducible. This decomposition when G_0 is Abelian is the subject of a recent interesting note of Godement.[7] It was this note of Godement together with a discussion of such a space for compact groups given by A. Weil[8] that suggested our definition of the Hilbert space H_L.

There are a number of questions suggested by the considerations of this note which we expect to investigate and report on at a later date. We close by mentioning a few of these. (1) When is the representation U of G generated by an irreducible representation L of G_0 itself irreducible? (2) When G is finite, L and U are finite dimensional and L is irreducible there is a classical theorem which says that the number of times that U contains a given irreducible representation V of G is equal to the number of times that the restriction of V to G_0 contains L. Weil[8] has recently extended this theorem to compact groups. One can ask whether (and in what sense) it continues to be true for general locally compact groups. (3) To what extent is it true that an arbitrary irreducible representation of G is the imprimitive representation generated by a primitive representation L of an appropriate G_0? How is the possible failure of this to hold generally connected with the "extra" representations of non-regular semidirect products? (4) Theorem 2 presumably can be used to prove other theorems like Theorem 3. What are some of these? One notes in particular that G_1 can probably be replaced by any group whose representations can be decomposed into irreducible parts in a suitably manageable manner.

* The author is a fellow of the John Simon Guggenheim Memorial Foundation on leave from Harvard University and in residence at the University of Chicago. A significant part of the work on this paper was done at each institution.

[1] Mackey, G. W., *Duke Math. J.*, **16**, 313–325 (1949).

[2] Cf. Nakano, H., *Ann. Math.*, **42**, 657–664 (1941), Hilfsatz 1.

[3] That such a measure always exists has been indicated by Dieudonné, *Ann. univ. Grenoble*, NS 23, 25–53 (1948) [p. 51]. Actually one can show, and this is important for our purposes, that any two quasi invariant measures have the same null sets. These null sets are precisely the sets whose inverse images in G are null sets with respect to Haar measure.

[4] In carrying through the details of this and other arguments indicated in this note it seems to be necessary to know that every Borel set in G which is a union of right G_0 cosets defines a Borel set in G/G_0. That this is so is an easy consequence of the following two theorems in the literature. A one-to-one continuous image of a Borel subset of a complete separable metric space is a Borel set. Kuratowski, C., *Topologie I*, Warsaw (1933), p. 251. If g is a continuous function on a compact subset of a metric space then there exists a Borel subset on which g is one-to-one and has the same range as before. Federer, H., and Morse, A. P., *Bull. Am. Math. Soc.*, **49**, 270–277 (1943) [Theorem 5.1].

[5] Gelfand, I., and Neumark, M., C. R. (Doklady), *Acad. Sci. Ukr. S. S. R.* (NS), **55,** 567–570 (1947).

[6] Wigner, E., *Ann. Math.*, **40,** 149–204 (1939).

[7] Godement, R., *C. R. Acad. Sci. Paris*, **228,** 627–628 (1949).

[8] Weil, A., *L'integration dans les groupes topologiques et ses applications*, Paris (1940), pp. 82, 83.

LINKAGE STUDIES OF THE RAT. X

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1. Fawn Linked with Agouti in Linkage Group V.—In an earlier paper of this series, VIII, 1947,[1] a new color gene, fawn, was described and designated by the symbol, *f*. Preliminary linkage tests gave only negative results, but a cross between Agouti and fawn made by Dr. King indicated possible linkage between the two genes. Additional crosses of this nature made by Castle in Berkeley fully support that hypothesis.

A non-agouti black fawn stock, *aaff*, was crossed with a gray race, *AAFF*. The F_1 animals were gray of genotype $\frac{af}{AF}$. They were back-crossed to the black fawn stock, yielding four classes of young, expected to be equal numerically, unless linkage exists between *A* and *f*. The four genotypes expected and their observed frequencies were as follows:

GRAY FAWN	BLACK FAWN	GRAY	BLACK
$\frac{Aaff}{245}$	$\frac{aaff}{310}$	$\frac{AaFf}{358}$	$\frac{aaFf}{294}$

The two back-cross classes, gray fawn and black, would derive from crossover (recombination) gametes produced by the F_1 parent, whereas black fawn and gray would derive from non-crossover gametes. Clearly the non-crossovers exceed the crossovers numerically, indicating linkage.

The crossover percentage is 44.6 ± 0.97, a highly significant statistical result.

We conclude that genes *A* and *f* lie in the same fifth chromosome of the rat. The linkage is loose but unmistakable.

Linkage systems previously demonstrated, as now understood are as follows:

I.	*p*	*r*	*c*	*l*	*w*
	0	20.5	21	24.3	66.3
II.	*Sh*	*Cu*	*an*	*in*	*b*
	0	4	14.3	28	49
III.	*hr*		*wo*		
	0		40.3		
IV.	*k*		*st*		
	0		34.1		

2. Further Evidence of Linkage Between Red-Eyed Yellow and Waltzing in the Rat.—In earlier papers evidence has been presented that linkage exists in the rat between albinism, pink-eyed yellow and waltzing, all of which genes lie in the first chromosome of the rat[2]. Since red-eyed yellow is linked with pink-eyed yellow and with albinism, it follows that linkage should also exist between red-eyed yellow and waltzing. An opportunity to test this hypothesis by experiment recently presented itself.

A cross was made between a red-eyed hooded waltzer and a gray hooded non-waltzer, all the F_1 young being as expected, hooded and non-waltzers. A backcross to red-eyed waltzers produced as expected both red-eyed and dark-eyed waltzers but in very unequal proportions.

It is known that waltzing has very low penetrance, i.e., waltzers are regularly produced in smaller numbers than expected. In the backcross under discussion, if waltzing is a simple recessive, 50% of the young should be waltzers. In reality the percentage is much lower. In a total of 390 back-cross young, only 45 were recognizable as waltzers, this being 11.6% whereas 50% was expected.

Of the 45 waltzers, 30 were red-eyed and 15 dark-eyed, the non-waltzers numbering 180 red-eyed and 165 dark-eyed. In other words, the incidence of waltzing among the red-eyed was 1 in 7, but among the dark-eyed it was 1 in 12. This supports the idea that waltzing is linked with red-eye as well as with pink-eye and albinism, since otherwise red-eyed and dark-eyed waltzers should occur with equal frequency. In reality they deviate significantly from the equality expected, if linkage does not exist. Deviation, 7.5/P.E., 2.26=3.3.

Acknowledgment.—Grateful thanks are due to the Division of Genetics, College of Agriculture, University of California, and particularly to Dr. E. R. Dempster for facilities which made this investigation possible.

[1] Castle, W. E., and King, Helen Dean, "Linkage Studies of the Rat. VIII. Fawn, a New Color Dilution Gene," *J. Heredity*, **38**, 343 (1947).

[2] Castle, W. E., "Linkage in the Albino Chromosome of the Rat," PROC. NATL. ACAD. SCI., **32**, 33–36 (1946).

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